On applications of the 3D-cranking model
to even-even systems with triaxiality and octupolarity

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Applications of the self-consistent three-dimensional cranking model are discussed in
the context of dynamical motions of triaxial and tetrahedral deformed states. A relation
between non-uniform rotation and symmetry breaking is discussed, particularly the role of
the tunnelling effect in finite systems.

§1. Introduction

Three-dimensional rotation was first suggested by Bohr and Mottelson in their
theory of nuclear wobbling motion,\textsuperscript{1)} which was finally observed in a recent experi-
ment on the Lu isotopes in the $A \approx 160$ mass region.\textsuperscript{2)} Realisation of the wobbling
motion requires two conditions: (i) very high spin ($J_1 \approx J \gg 1$, and equivalently
$J_2, J_3 \ll 1$) and (ii) the presence of triaxial deformation. In principle, this wob-
bling motion is attributed to the kinematics of a macroscopic rotor with triaxial
deformation. The corresponding classical Hamiltonian reads,

$$H = \sum_{i=1}^{3} \frac{J_{i}^2}{2\mathcal{I}_i} = \frac{J_1^2}{2\mathcal{I}_1} + \frac{J_2^2}{2\mathcal{I}_2} + \frac{J_3^2}{2\mathcal{I}_3},$$

(1.1)

where the second term is given as

$$\epsilon(J_2, J_3) = \left(\frac{1}{\mathcal{I}_2} - \frac{1}{\mathcal{I}_1}\right) J_2^2 + \left(\frac{1}{\mathcal{I}_3} - \frac{1}{\mathcal{I}_1}\right) J_3^2.$$

(1.2)

After the quantisation procedure, the perturbation term $\frac{1}{2}\epsilon(J_2, J_3)$ turns out to be
$\hbar \omega (n + \frac{1}{2})$, which has the same form as the quantised energy levels of a harmonic
oscillator. In this way, the wobbling motion can be considered as a perturbation
against the one-dimensional rapid rotation of the rigid rotor. In other words, the
motion can be considered almost as 1D rotation, although it is surely 3D rotation
with small amplitude oscillation of the total angular momentum vector around the
1st axis on which most of the angular momentum component is concentrated.

With the recent development in the tilted-axis cranking model, new types of
3D rotation induced by not only macroscopic but also microscopic effects have been
presented, such as magnetic rotation produced by anisotropy of the electric current;
tilted rotation created by breaking the signature symmetry; and chiral rotation as
a result of triaxial deformation and chiral symmetry breaking.\textsuperscript{3,4)} Based on the
theories of Kerman-Onishi\textsuperscript{5)} and Onishi-Horibata,\textsuperscript{6,7)} we have been investigating
high-spin phenomena related to these new types of 3D rotation from a microscopic,
self-consistent, and quantum mechanical basis.\textsuperscript{8)}
The aim of this paper is to report our attempt to find chiral solutions through a fully self-consistent treatment (3D-cranked HFB) of one of the Ce isotopes in the \( A \approx 130 \) mass region. Next, we briefly discuss and suggest possible 3D rotations of the tetrahedral deformed states, which were recently predicted theoretically by Dudek and his collaborators in Zr isotopes around \( A \approx 100 \).

§2. Two types of discrete symmetry

Many of the point-group symmetries are subgroups of the rotational group \( O(3) \). Such symmetries are seen not only in solid-state physics, but also in nuclear physics, such as signature (\( C_{2v} \)), triaxiality (\( D_2 \)), and tetrahedrality (Td). On the other hand, symmetries of parity, mirror reflection and time reversal belong to discrete symmetries that cannot be expressed by rotation. In this paper, we concentrate on (i) two kinds of point-group symmetry (\( D_2 \) and Td), and (ii) mirror reflection symmetry (or chiral symmetry).

Operations corresponding to these two types of discrete symmetry are unified by the unitary transformation. Matrix representations for these unitary transformations are uniquely classified by the determinant: when it is positive (negative) unity, the operation is called “proper” (“improper”).

In a finite system like a nucleus, rotations are closely connected to a “shape” of the whole system. The point group symmetry thus describes the symmetry breaking in the mean field, or deformation. In particular, essentially static characters of the finite system can be investigated by this type of discrete symmetry.

As seen with parity and chiral symmetry in particle physics, there are many cases where the improper unitary transformation gives us an insight about the dynamics of physical systems. In fact, the nuclear chiral configuration is of more dynamical character than the static chiral configurations seen in optical isomers in chemistry. This is because the nuclear chiral symmetry originates from the configuration of angular momentum vectors produced by a triaxially deformed core and valence nucleons (several proton particles and neutron holes, for instance), where the interplay among the valence particles and the rest of the system plays a significant role.

§3. Nuclear chirality

The original model for the chiral configuration is based on the presence of a valence particle, a valence hole, and a triaxial rotor whose angular momentum vectors point in different directions along the different principal axes of the core. Under the assumption of an irrotational liquid drop for a nucleus, the moment of inertia about the intermediate principal axis takes the maximum value when triaxiality is maximised (i.e., \( \gamma = 30^\circ \)). From the classical analogue, the collective rotation of the core is thus expected to be exclusively around the intermediate axis. The valence particles, on the contrary, should move around the shortest principal axis because

\footnote{Quadrupole deformation parameters for measures of triaxiality \( \gamma \) and elongation \( \beta \) are defined as the Hill-Wheeler coordinate, given on p. 7 of Ref.}
the overlap of the total and single-particle wave functions is maximised in this case. The valence holes should travel around the longest axis since particles and holes have opposite characters in their wave functions. For this reason, odd-odd systems, e.g., systems of a proton particle a neutron hole in the valence orbitals, and an even-even core are typical situations to realise the chiral rotation.

§4. Chiral rotations produced by quasiparticles

In spite of the preference of odd-odd systems for the realisation of nuclear chirality, there is a possibility for the emergence of nuclear chirality in even-even systems, if Cooper pairs can be broken simultaneously for both protons and neutrons. Rotation-aligned states with multi-quasiparticles can be natural candidates for chiral states in even-even systems if triaxial deformation is present.

According to Strutinski calculations,\textsuperscript{11} cerium isotopes around the $A \simeq 130$ region are believed to be gamma-soft in the ground state. It is empirically known that the gamma deformation becomes more robust at high spin, so that these isotopes are good candidates to have chiral configurations. Besides, it is experimentally observed that there are two types of excited bands, which are identified as aligned states of either protons or neutrons. In this way, cerium isotopes in the $A \simeq 130$ mass region are expected to have triaxial deformation as well as the simultaneous alignment of four quasiparticles (two protons and two neutrons).

§5. The three-dimensional cranking model

Kerman and Onishi developed the three-dimensional cranking model based on the Hartree-Fock theory.\textsuperscript{5} Horibata and Onishi later improved the model based on the Hartree-Fock-Bogoliubov (HFB) theory.\textsuperscript{7} This 3D-cranking model is microscopic and fully self-consistent. The numerical calculations were performed by the group led by Onishi, using the pairing-plus-$Q \cdot Q$ separable interaction. The present calculations are based on the 3D-cranked HFB theory, which will be briefly explained below.

The HFB ansatz $\varphi$ has a form of the general product function of the quasiparticle annihilation operator $\beta_i$,

$$|\varphi\rangle = \prod_i \beta_i |0\rangle,$$

where $|0\rangle$ denotes the vacuum in the Fock space defined with respect to the canonical basis $\{a_i, a_i^\dagger\}$. In our calculation, the spherical Nilsson basis is employed for the canonical basis. The HFB ansatz corresponds to the vacuum in the quasiparticle space:

$$\beta_i |\varphi\rangle = 0.$$  \hspace{1cm} (5.2)

The canonical basis and quasiparticle basis are related by the unitary transformation called the general Bogoliubov transformation,

$$\begin{pmatrix} a_i^\dagger \\ a_i \end{pmatrix} = \begin{pmatrix} U_{im} & V_{im}^* \\ V_{im} & U_{im}^* \end{pmatrix} \begin{pmatrix} \beta_m^\dagger \\ \beta_m \end{pmatrix}.$$  \hspace{1cm} (5.3)
The HFB ansatz is now parametrised by $U_{im}$ and $V_{im}$, and they are determined by the variational equation with constraints,

$$\delta \langle \varphi | \hat{H} - \sum_{\tau=p,n} \lambda_{\tau} \hat{N}_{\tau} - \sum_{i=1}^{3} \omega_{i} \hat{J}_{i} - \sum_{i=1}^{3} \mu_{i} \hat{B}_{i} | \varphi \rangle = 0,$$  

(5.4)

with

$$\langle \hat{N}_{\tau} \rangle = N_{\tau}, \quad \langle \hat{J}_{i} \rangle = J_{i},$$

(5.5)

$$\langle \hat{B}_{k} \rangle = \frac{1}{2} \langle (\hat{Q}_{ij} + \hat{Q}_{ji}) \rangle = 0 \quad (i, j \text{ and } k; \text{ cyclic})$$

(5.6)

The Hamiltonian has a form,

$$\hat{H} = \sum_{i} \epsilon_{i} a_{i}^{\dagger} a_{i} + \hat{V},$$

(5.7)

where the residual interaction $\hat{V}$ is expressed,

$$\hat{V} = -\frac{X}{2} \sum_{m=-2}^{2} : \hat{Q}_{m}^{\dagger} \cdot \hat{Q}_{m} : - \sum_{\tau} G_{\tau} \hat{P}_{\tau}^{\dagger} \hat{P}_{\tau},$$

(5.8)

where the quadrupole operator $\hat{Q}_{m}$ and the creation operator for a Cooper pair $\hat{P}_{\tau}$ are defined respectively as,

$$\hat{Q}_{m} = \sum_{kk'} \langle k | p^{2} Y_{2m} | k' \rangle a_{k}^{\dagger} a_{k'}, \quad \hat{P}_{\tau} = \sum_{k>0} a_{k}^{\dagger} a_{k}^{\dagger}.$$  

(5.9)

The symbol “: … :” in Eq. (5.8) expresses the normal ordering defined on p.601 of Ref.\textsuperscript{10} A prescription for the model space to diagonalise $\hat{V}$ is employed from the papers of Kumar and Baranger (i.e., two major shells for each isospin sector).\textsuperscript{12} The force parameter is determined in the standard way of the pairing-plus-$Q \cdot Q$ interaction model. The HFB equation is solved by means of the method of steepest descent. Details of the method can be found in Ref.\textsuperscript{7}

§6. Results of 3D cranked HFB calculations

In this paper, we concentrate on a state at $J = 30 \hbar$ whose energy surface is shown in Fig.1. Detailed discussions about other states were presented in Ref.\textsuperscript{13} There are minima labelled as “is-TAR” in the figure such as $(\theta, \phi) = (90^\circ, 30^\circ)$, which means “tilted axis rotation in the $i$-$s$ plane”. The $i$-$s$ plane is spanned by the intermediate and shortest principal axes of the core. This minimum corresponds to two-dimensional rotation. A minimum labelled as “PAR” is a state of principal axis rotation, i.e., one-dimensional rotation. The two minima of is-TAR as well as two of the minima of PAR (except $(\theta, \phi) = (90^\circ, 90^\circ)$) have the same intrinsic structure, respectively. Below, only the minima at $\theta = 90^\circ$ both for PAR and is-TAR are considered. An analysis of the PAR minimum at $(\theta, \phi) = (90^\circ, 90^\circ)$ is omitted here. See Ref.\textsuperscript{13}
The energy surface along $\theta = 90^\circ$ (for $|\phi| < 45^\circ$) has a structure similar to a “double well potential”. (See Fig.2.) Classically, the is-TAR becomes the ground state as a result of spontaneous breaking of the signature symmetry. Quantally, however, the tunnelling effect can connect $|R\rangle = |\phi = +30^\circ\rangle$ and $|L\rangle = |\phi = -30^\circ\rangle$ to give rise to symmetrised ($|S\rangle$) and anti-symmetrised ($|A\rangle$) states. In other words,

$$|S\rangle \propto |L\rangle + |R\rangle, \quad \text{and} \quad |A\rangle \propto |L\rangle - |R\rangle.$$

This tunnelling can be interpreted as a dynamical motion oscillating between these local minima of is-TAR. The calculations show the deformation is almost constant for arbitrary $\phi$ along $\theta = 90^\circ$, and the oscillating direction is mainly confined along the $\phi$ direction. From these facts, this dynamical motion is similar to “nutation” of a classical rigid body under rotation. A schematic picture of the nutation is presented in the upper part of Fig.2.

There is another interesting minimum at $(\theta, \phi) \simeq (60^\circ, 45^\circ)$. At this point, triaxiality is calculated to be $\gamma = 60^\circ$, that is, the shape is oblate (axially symmetric). This solution should be thus interpreted as tilted rotation of the oblate deformed state. There is a structure around this minimum looking like a shallow valley. The corresponding dynamical motion is expected to be collective rotation around the symmetry axis, with the tilt angle between the symmetry axis and the total spin vector being kept fixed to the values at $(\theta, \phi) \simeq (60^\circ, 45^\circ)$. From the shallowness of the energy surface, it is evident that this collective rotation requires nearly no energy, which implies a zero mode. When the deviation of the $\gamma$ from value $60^\circ$ becomes substantial, this zero mode collapses (or “decays”) into the nutating mode of triaxial deformation ($\gamma \simeq 30^\circ$). A schematic picture is presented in Fig. 2.

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**Fig. 1.** Energy surface for $^{134}$Ce at $J = 30h$. 

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§7. An analogy with the linear $\sigma$ model

The nutation and the zero mode in the last section present us a good example of spontaneous symmetry breaking in nuclear systems with quadrupole deformation. It should be noted here that a physical phenomenon like nutation is a unique characteristic of nuclear spontaneous symmetry breaking, which cannot be seen in other physical systems with infinite degrees of freedom. The aim of this section is to clarify similarities and differences in spontaneous symmetry breaking between systems with finite and infinite degrees of freedom. For this aim, let us consider the simplest case from field theory, that is, the linear $\sigma$ model.\textsuperscript{14}

The Lagrangian is defined as

$$L = \frac{1}{2} (\partial_\mu \phi^j)^2 - V(\{\phi^j\}), \quad (7.1)$$

where the potential is given by

$$V(\{\phi^j\}) \equiv \frac{1}{2} m^2 (\phi^j)^2 + \frac{\lambda}{4} \left[(\phi^j)^2\right]^2. \quad (7.2)$$

The index $j$ runs from 1 to $N$, and follows Einstein’s convention. This Lagrangian possesses the symmetry $O(N)$. The sign of the “mass parameter” $m^2$ controls the absence and presence of spontaneous symmetry breaking. In the case when $m^2 < 0$, the vacuum expectation value can be non-trivial, and is given as the local minimum of the potential, i.e., $\frac{\partial V(\phi)}{\partial \phi} = 0$. Although there are infinite numbers of degenerate vacua that are connected to each other by the $O(N)$ operation, we can choose one of the vacuum expectation values to be $\langle \phi \rangle = (0, 0, \ldots, v)$ with $v = \frac{M}{\sqrt{\lambda}}$, by putting $m^2 = -M^2$. A set of new fields are introduced in order to describe the fluctuation around the vacuum, that is, $\phi = (\Pi^1(x), \Pi^2(x), \ldots, \Pi^{N-1}(x), v + \sigma(x))$ Now the Lagrangian is rewritten as

$$L = \frac{1}{2} (\partial_\mu \sigma)^2 - \frac{1}{2} (2M^2)\sigma^2 - \sqrt{\lambda} M \sigma^3 - \frac{\lambda}{4} \sigma^4$$

$$+ \frac{1}{2} (\partial_\mu \Pi^i)^2 - \frac{\lambda}{4} [(\Pi^i)^2]$$

$$- \sqrt{\lambda} M (\Pi^i)^2 \sigma - \frac{\lambda}{2} (\Pi^i)^2 \sigma^2. \quad (7.3)$$

The symmetry of the Lagrangian is now reduced to $O(N-1)$ possessed by $\Pi^i$ massless fields. The $\sigma$ field is massive with its mass being $\sqrt{2} M$.

When these fields are quantised, quantum fluctuations are expected around the vacuum. Along the $\sigma$ degree of freedom, only small oscillation is possible because the vacuum sits in the local minimum (say, “the bottom of the potential valley”). It is important to recall that there is no tunnelling in infinite systems. Suppose there could be tunnelling between a state of $\phi^N(x) = v$ and $\phi^N(x) = -v$, and that the tunnelling probability could be given as $p(< 1)$ equally for all space-time points $x$. If the number of degrees of freedom in the space-time coordinates was finite and given by $N$, then the total probability for tunnelling in all the degrees of freedom would
be \( p^N \), which disappears when \( N \rightarrow \infty \). Therefore, there is no need to consider the tunnelling in the infinite systems. On the other hand, along the \( \Pi^i \) directions, large amplitude dynamical modes are possible. The curvature of the potential is completely “flat” along these degrees of freedom, so that the tunnelling plays no role. This character of the \( \Pi^i \) fields can be easily understood because they correspond to massless particles, like photons having an infinite range as a quantised mediator of the electromagnetic force. They are created as a result of the spontaneous breaking of continuous symmetries, and are called Goldstone bosons (or Nambu-Goldstone bosons).

The Goldstone bosons are similar to the zero modes in a nuclear system, which correspond to collective rotations. For example, like the Goldstone bosons, collective rotation occurs when the continuous symmetries, such as \( O(3) \) and \( O(2) \), are broken spontaneously by a deformed mean field. In addition, the collective motion can be understood as a large amplitude motion, as the Goldstone bosons have long correlation length owing to their being massless.

The motion of the nuclear nutation looks similar to the massive \( \sigma \) field as long as the oscillation amplitude of the nutation is small enough. However, because of the tunnelling in nuclear systems, large amplitude nutation is possible. In other words, in finite systems, tunnelling has a role to restore the spontaneously broken symmetries, like the Goldstone boson. This point is unique in the finite systems, and can be very important when we try to understand the dynamics of nuclei and other finite systems.

§8. Possible dynamical motions of a tetrahedral deformed state

In the study of \(^{16}\text{O}\), which is spherical in the shell model, Brink showed the possibility of tetrahedral deformation based on the alpha clustering model.\(^{15}\) The dynamics of the tetrahedral alpha clustering structure was investigated by Onishi-Sheline.\(^{16}\) Takami et al., investigated this exotic shape in heavier systems by means of the mean-field theory (Skyrme HF + BCS).\(^{17}\) Dudek et al. also studied single-particle level structure influenced by tetrahedral symmetry in a nuclear mean-field in the medium mass region.\(^{9}\)

Dudek suggests a certain rotational spectrum pattern as experimental evidence of tetrahedral deformation. However, his analysis is based on the quantum rotor without internal structure, assuming the tetrahedral shape a priori. As he pointed out, the shell structure plays an important role in sustaining the tetrahedral shape. It is thus necessary to study the system from a microscopic viewpoint.

By analogy with the dynamics of \(^{16}\text{O}\) and tetrahedrality, a variety of vibrational and rotational modes are expected due to the point-group symmetry \( T_d \). Some of the modes are displayed in Fig.3. In future, we plan to contribute to this research field through performing the 3D-cranked HFB calculations as well as the Skyrme-HF+BCS calculations.
Fig. 2. Tunnelling and Goldstone mode in the $J = 30h$ state of $^{134}\text{Ce}$.

Fig. 3. Possible vibrations expected in the tetrahedral deformed mean fields.

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References

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