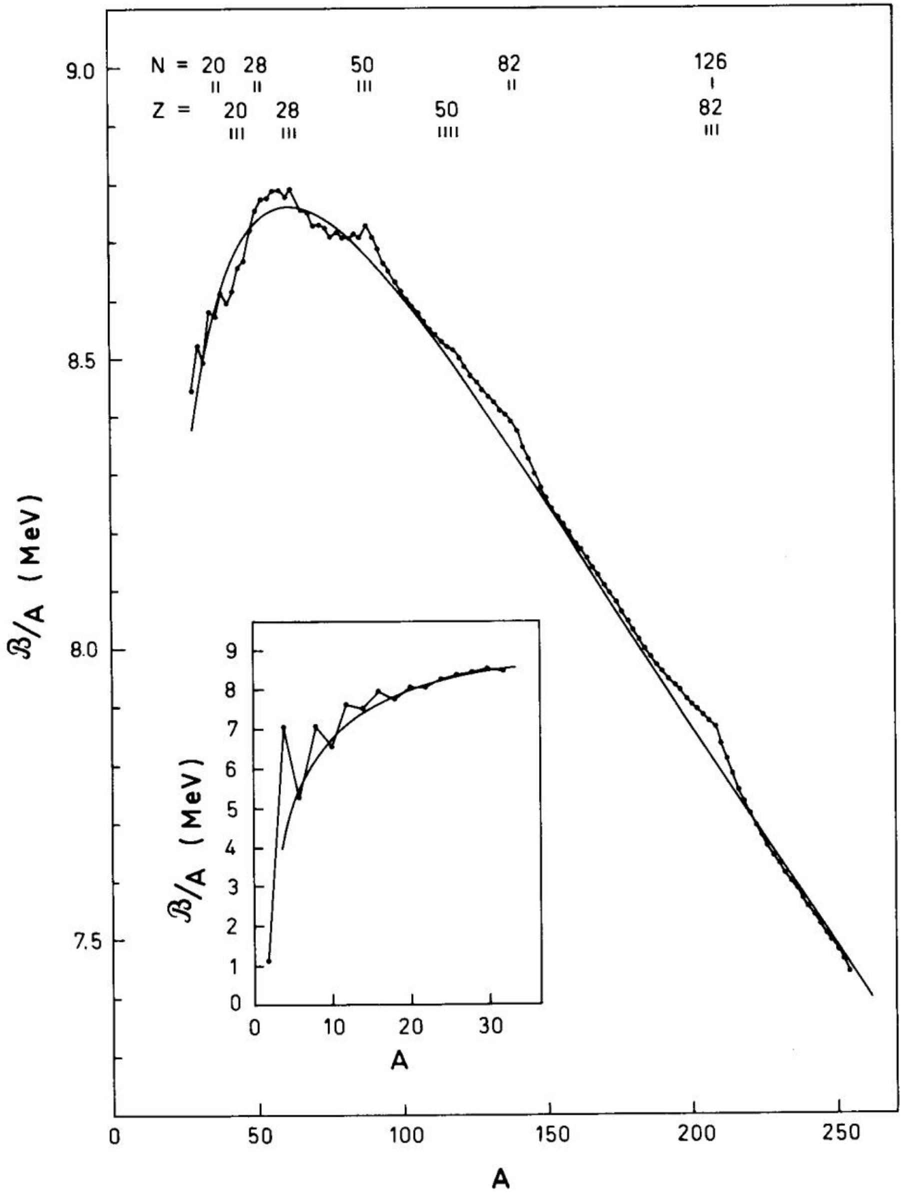
Shell Structure

$$B(N,Z) = B_{\text{macro}}(N,Z) + B_{\text{micro}}(N,Z)$$



- Smooth part

$$B_{\text{macro}}(N,Z) = a_v A - a_s A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_{\text{sym}} \frac{(N-Z)^2}{A}$$

- Fluctuation part

$$B_{\text{micro}} = B_{\text{pair}} + B_{\text{shell}}$$

Liquid drop model:

$$B_{\text{LDM}} = B_{\text{macro}} + B_{\text{pair}}$$

Pairing Energy

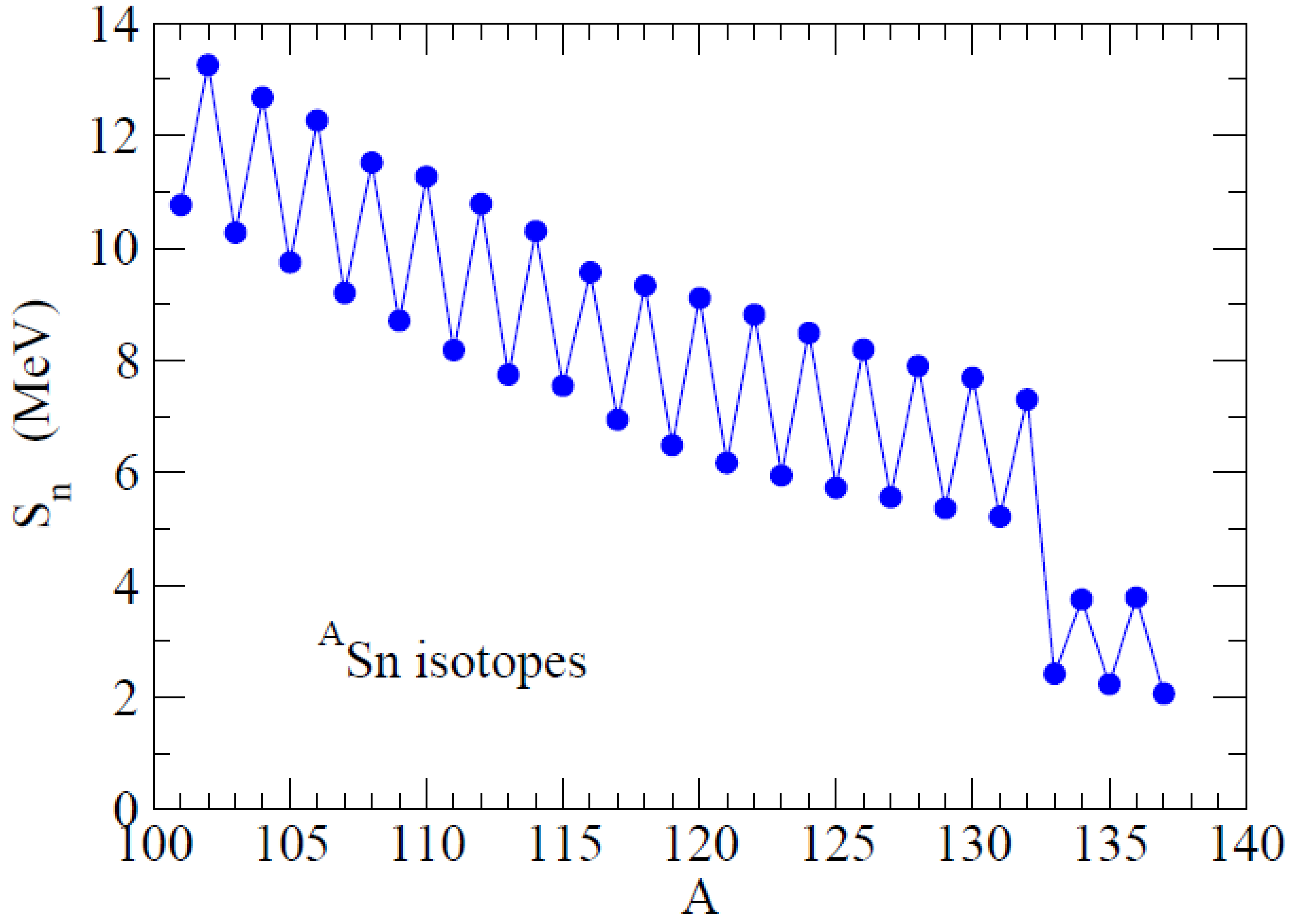
Extra binding when like nucleons form a spin-zero pair

Example:

	Binding energy (MeV)
${}^{210}_{82}\mathrm{Pb}_{128} = {}^{208}_{82}\mathrm{Pb}_{126}+2\mathrm{n}$	1646.6
${}^{210}_{83}\mathrm{Bi}_{127} = {}^{208}_{82}\mathrm{Pb}_{126}+\mathrm{n}+\mathrm{p}$	1644.8
${}^{209}_{82}\mathrm{Pb}_{127} = {}^{208}_{82}\mathrm{Pb}_{126}+\mathrm{n}$	1640.4
${}^{209}_{83}\mathrm{Bi}_{126} = {}^{208}_{82}\mathrm{Pb}_{126}+\mathrm{p}$	1640.2

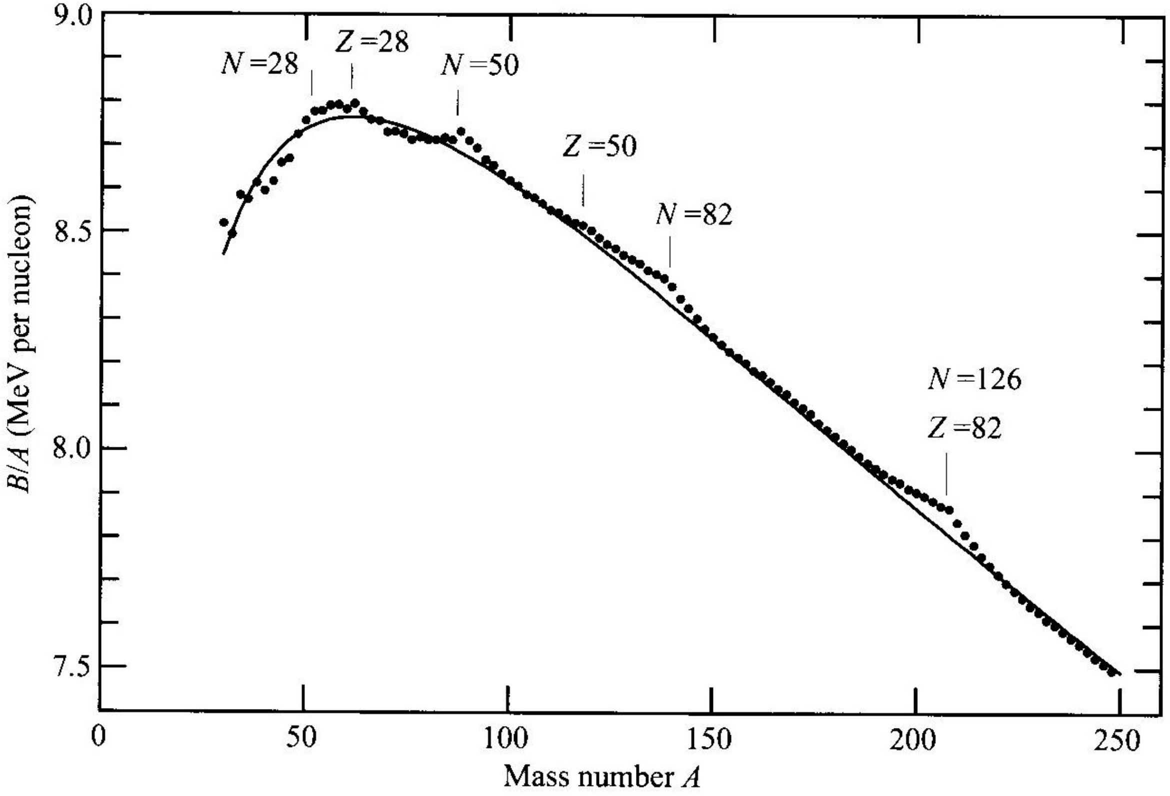
$$\boxed{\begin{aligned} B_{\mathsf{pair}} &= \Delta && (\mathsf{for\ even-even}) \\ &= 0 && (\mathsf{for\ even-odd}) \\ &= -\Delta && (\mathsf{for\ odd-odd}) \end{aligned}}$$

even-odd staggering



1n separation energy: S_n (A,Z) = B(A,Z) – B(A-1,Z)

Shell Energy



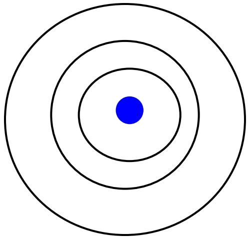
Extra binding for N,Z = 2, 8, 20, 28, 50, 82, 126 (magic numbers)

⇨ Very stable

${}^{4}_{2}He_{2}$, ${}^{16}_{8}O_{8}$, ${}^{40}_{20}Ca_{20}$, ${}^{48}_{20}Ca_{28}$, ${}^{208}_{82}Pb_{126}$

(note) Atomic magic numbers (Noble gas)

He (Z=2), Ne (Z=10), Ar (Z=18), Kr (Z=36), Xe (Z=54), Rn (Z=86)

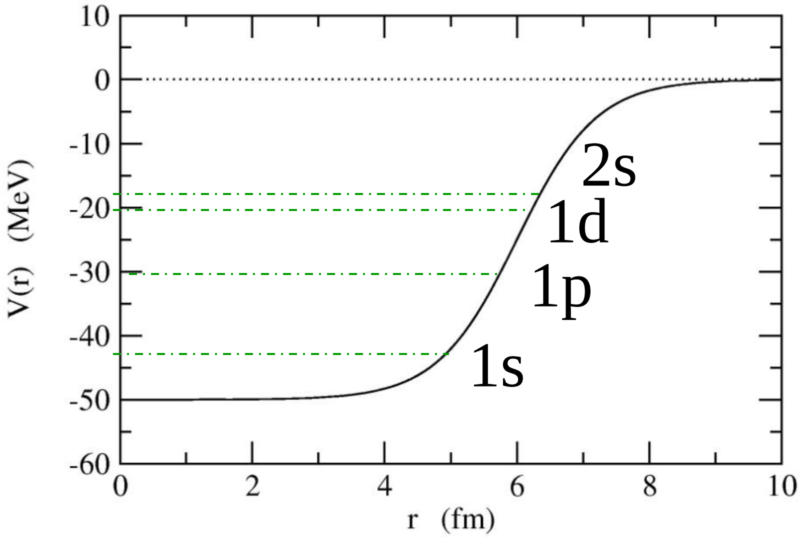


Shell structure

Similar attempt in nuclear physics: independent particle motion in a potential well

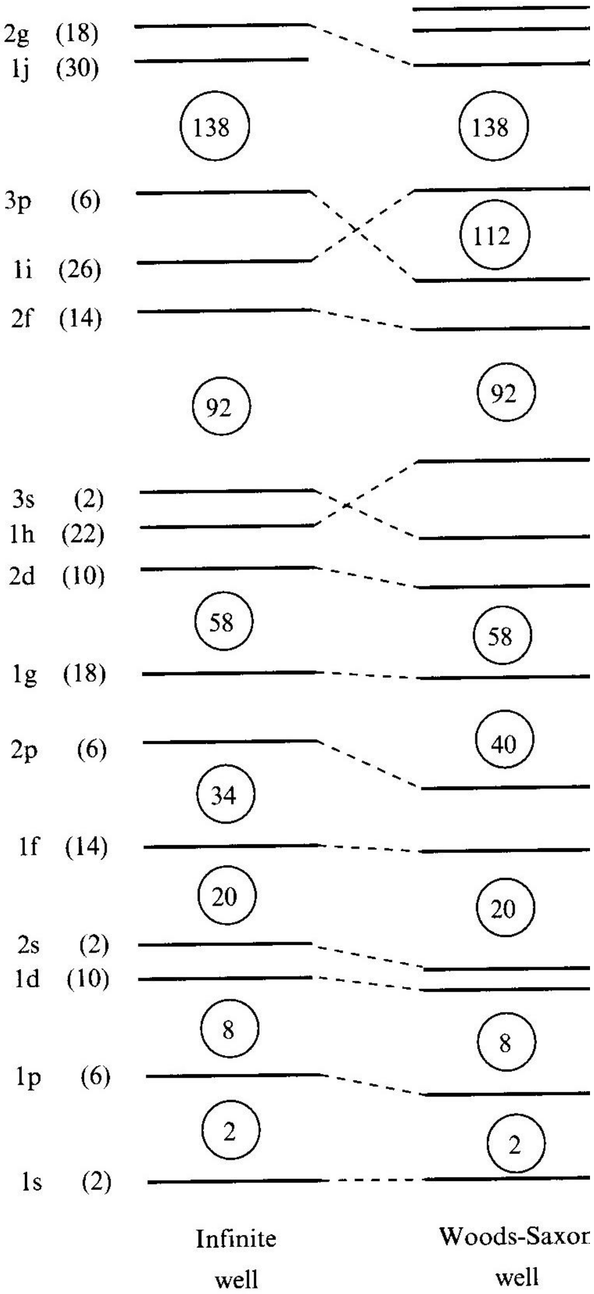
Woods-Saxon potential

$$V(r) = -V_0/[1 + \exp((r - R_0)/a]$$



$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) - \epsilon\right]\psi(\boldsymbol{r}) = 0$$

$$\psi(\boldsymbol{r}) = \frac{u_l(r)}{r} Y_{lm}(\hat{\boldsymbol{r}}) \cdot \chi_{m_s}$$



Woods-Saxon itself does not provide the correct magic numbers (2,8,20,28, 50,82,126).

Meyer and Jensen (1949):

Strong spin-orbit interaction

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) + \boxed{V_{ls}(r)\boldsymbol{l}\cdot\boldsymbol{s}} - \epsilon\right]\psi(\boldsymbol{r}) = 0$$

$$V_{ls}(r) \sim -\lambda\frac{1}{r}\frac{dV}{dr} \qquad (\lambda \ > \ 0)$$

jj coupling shell model

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) - \epsilon\right]\psi(\boldsymbol{r}) = 0 \quad \Longrightarrow \quad \psi_{lmm_s}(\boldsymbol{r}) = \frac{u_l(r)}{r} Y_{lm}(\hat{\boldsymbol{r}}) \cdot \chi_{m_s}$$

Spin-orbit interaction

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) + V_{ls}(r)\boldsymbol{l}\cdot\boldsymbol{s} - \epsilon\right]\psi(\boldsymbol{r}) = 0$$

(note) $\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s} \quad \Longrightarrow \quad \boldsymbol{l}\cdot\boldsymbol{s} = (\boldsymbol{j}^2 - \boldsymbol{l}^2 - \boldsymbol{s}^2)/2$

$$\boxed{\begin{aligned} \psi_{jlm}(\boldsymbol{r}) &= \frac{u_{jl}(r)}{r}\mathcal{Y}_{jlm}(\hat{\boldsymbol{r}}) \\ \mathcal{Y}_{jlm}(\hat{\boldsymbol{r}}) &= \sum_{m_l, m_s} \langle l\ m_l\ 1/2\ m_s | j\ m\rangle Y_{lm_l}(\hat{\boldsymbol{r}})\chi_{m_s} \end{aligned}}$$

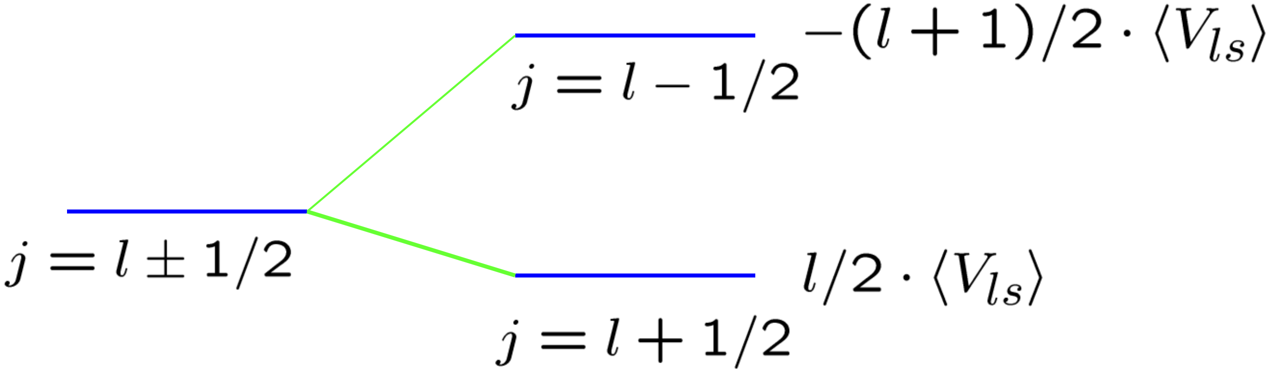
jj coupling shell model

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) + V_{ls}(r)\boldsymbol{l}\cdot\boldsymbol{s} - \epsilon\right]\psi(\boldsymbol{r}) = 0$$

(note) $\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s} \quad \Longrightarrow \quad \boldsymbol{l}\cdot\boldsymbol{s} = (\boldsymbol{j}^2 - \boldsymbol{l}^2 - \boldsymbol{s}^2)/2$

$$\begin{aligned} \psi_{jlm}(\boldsymbol{r}) &= \frac{u_{jl}(r)}{r}\mathcal{Y}_{jlm}(\hat{\boldsymbol{r}}) \\ \mathcal{Y}_{jlm}(\hat{\boldsymbol{r}}) &= \sum_{m_l,m_s} \langle l \ m_l \ 1/2 \ m_s | j \ m\rangle Y_{lm_l}(\hat{\boldsymbol{r}})\chi_{m_s} \end{aligned}$$

$$\boldsymbol{l}\cdot\boldsymbol{s} = l/2 \ (j = l + 1/2), \quad -(l+1)/2 \ (j = l - 1/2)$$



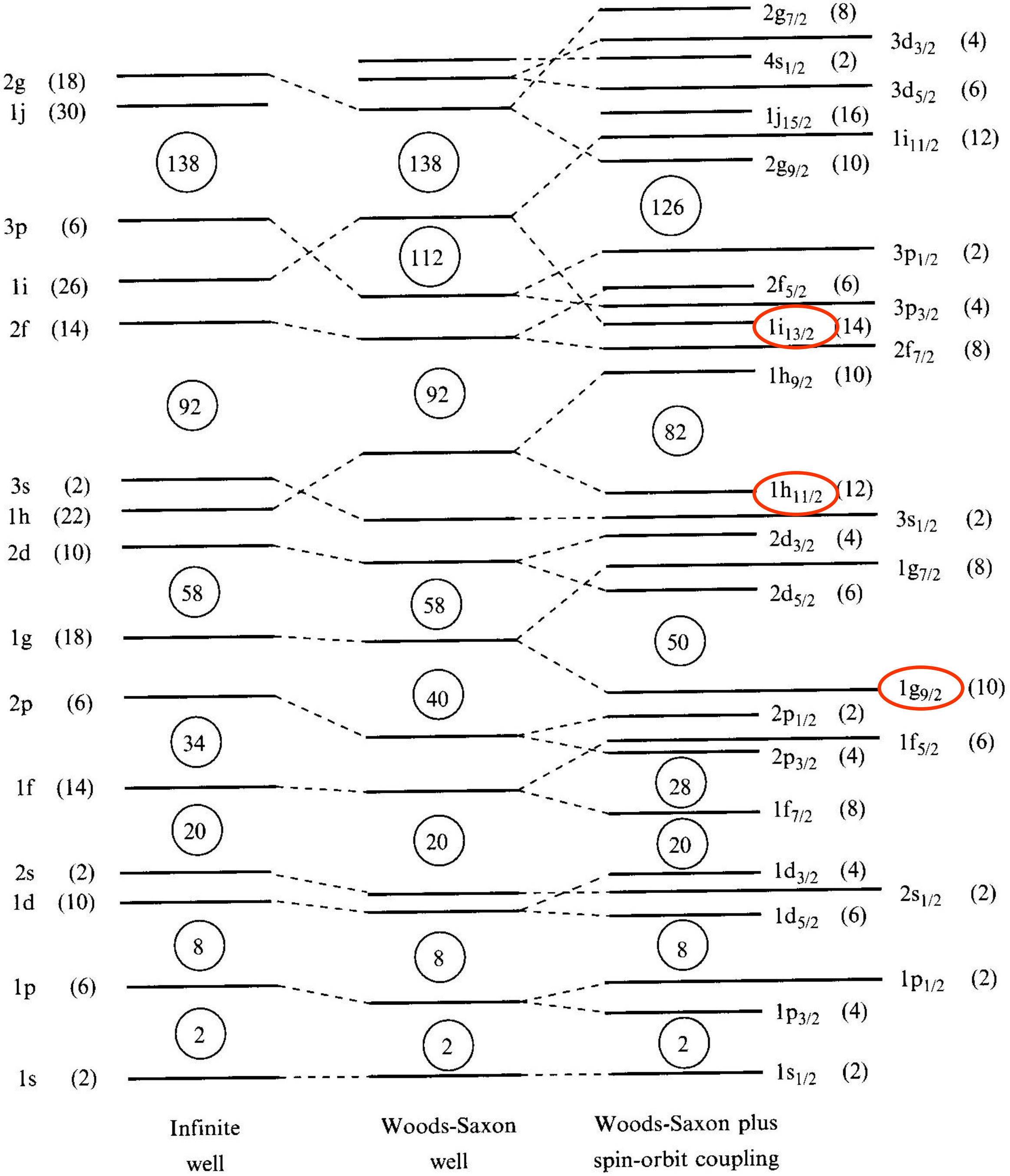
$-(l+1)/2 \cdot \langle V_{ls}\rangle$

$j = l - 1/2$

$j = l \pm 1/2$

$l/2 \cdot \langle V_{ls}\rangle$

$j = l + 1/2$



intruder states

unique parity states

Single particle spectra

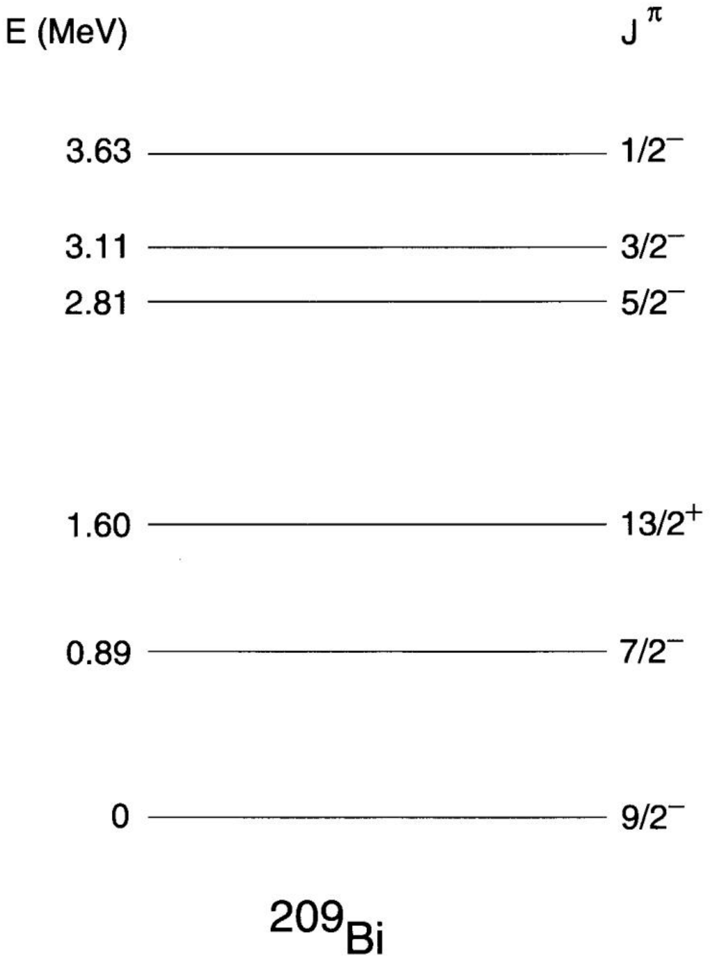
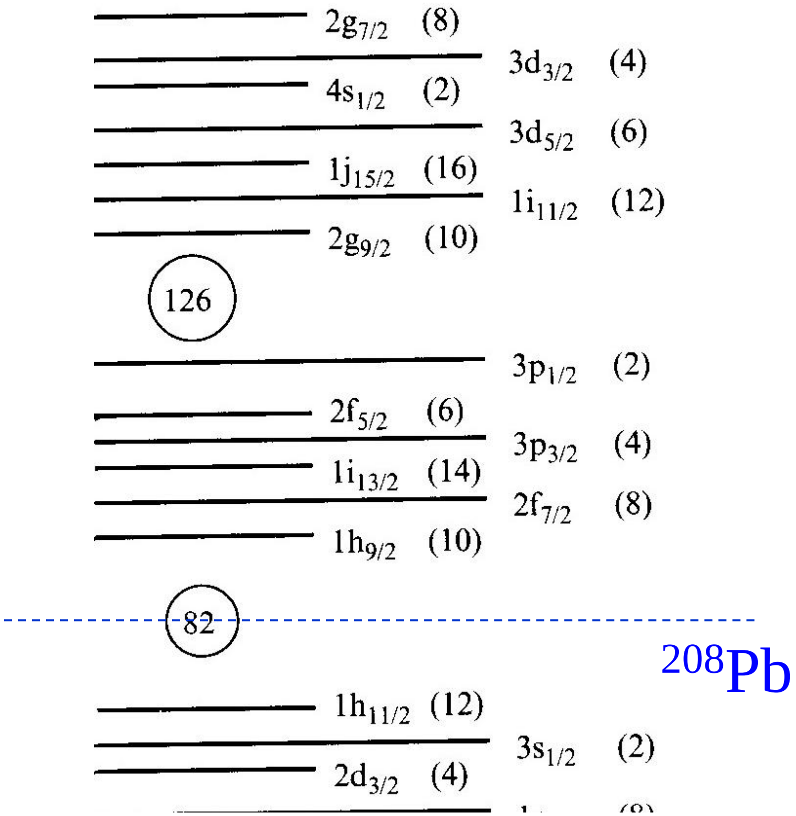


FIG. 3.6. Low-lying single-particle levels of ^{209}Bi.



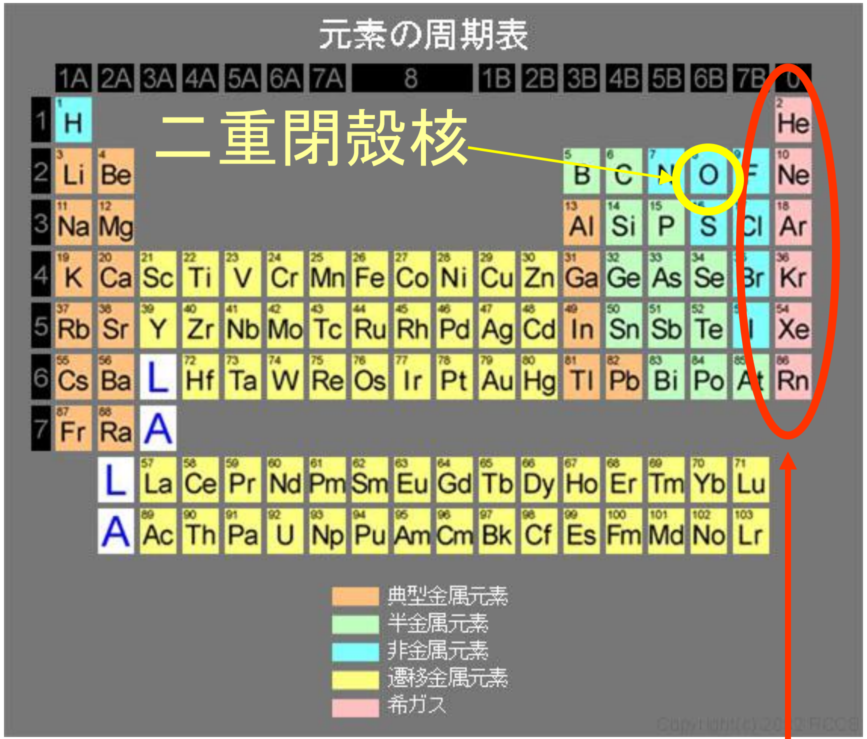
- How to construct V(r) microscopically?
- Does the independent particle picture really hold?

⇨ Later in this lecture

生命誕生のための幸運な偶然

原子の魔法数

電子の数が 2, 10, 18, 36, 54, 86



元素の周期表

	1A	2A	3A	4A	5A	6A	7A	8			1B	2B	3B	4B	5B	6B	7B	0
1	1 H																	2 He
2	3 Li	4 Be											5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg											13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	L	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	A															
		L	57 La	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu	
		A	89 Ac	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr	

- 🟧 典型金属元素
- 🟩 半金属元素
- 🟦 非金属元素
- 🟨 遷移金属元素
- 🟥 希ガス

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不活性ガス：He, Ne, Ar, Kr, Xe, Rn

原子核の魔法数

陽子または中性子の数が
2, 8, 20, 28, 50, 82, 126 の時安定

➡ 例えば $^{16}_{8}\mathrm{O}_{8}$（二重閉殻核）

➡ 酸素元素は元素合成の過程で数多く生成された

➡ しかし、酸素は化学的には「活性」

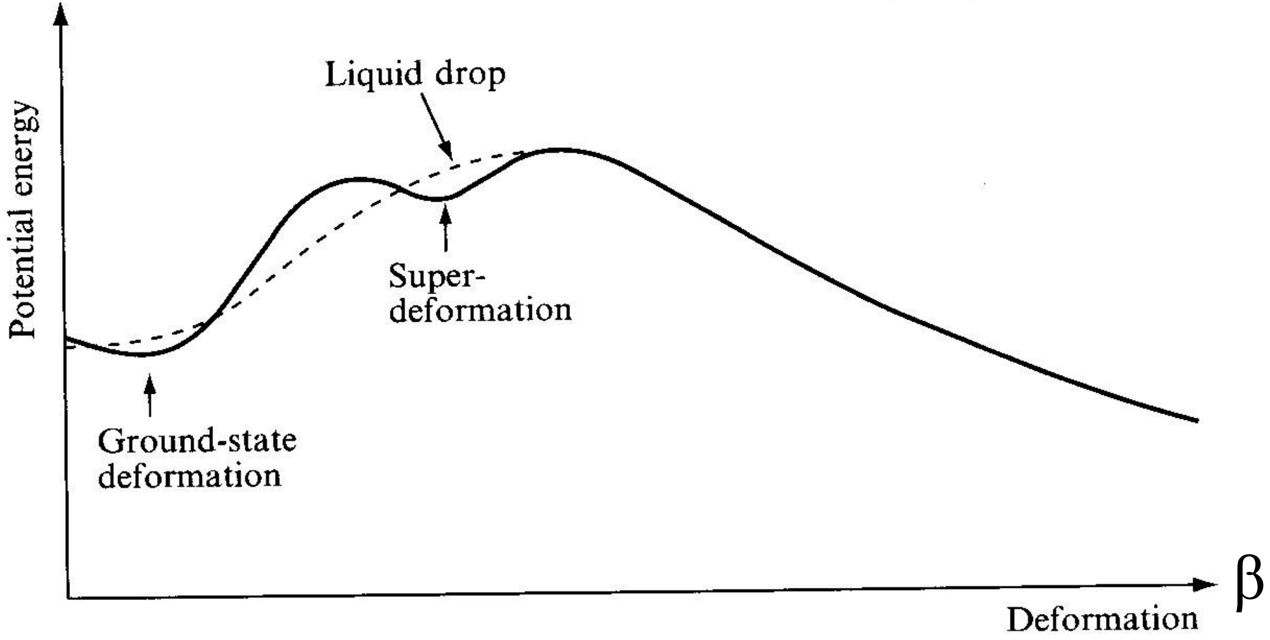
➡ 化学反応により様々な複雑な物質をつくり生命に至った

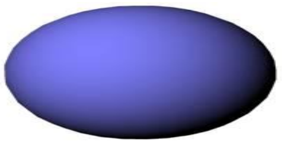
参考：望月優子　ビデオ「元素誕生の謎にせまる」　http://rarfaxp.riken.go.jp/~motizuki/contents/genso.html

Nuclear Deformation

Deformed energy surface for a given nucleus

$$E(\beta) = E_{LDM}(\beta) + E_{\mathsf{shell}}(\beta)$$





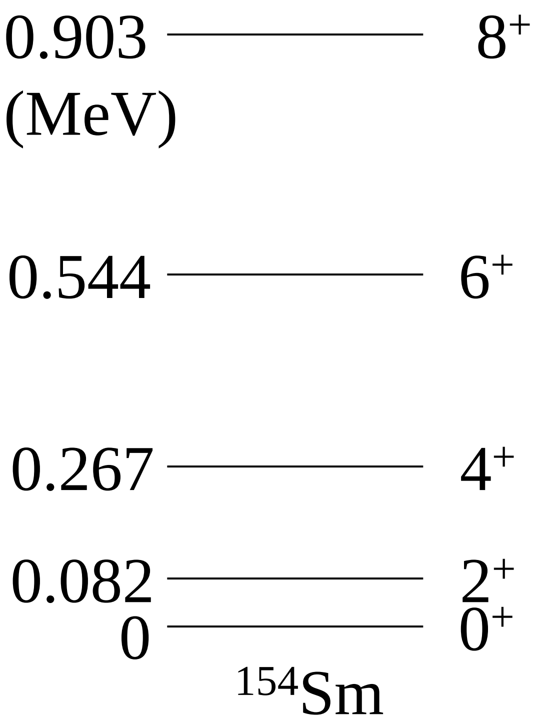
LDM only → always spherical ground state

Shell correction → may lead to a deformed g.s.

* Spontaneous Symmetry Breaking

Nuclear Deformation

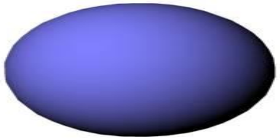
Excitation spectra of ^{154}Sm



0.903 (MeV)		8^+
0.544		6^+
0.267		4^+
0.082		2^+
0		0^+

^{154}Sm

$$\boxed{E_I \sim \frac{I(I+1)\hbar^2}{2\mathcal{J}}}$$



cf. Rotational energy of a rigid body
(Classical mechanics)

$$E = \frac{1}{2}\mathcal{J}\omega^2 = \frac{I^2}{2\mathcal{J}}$$

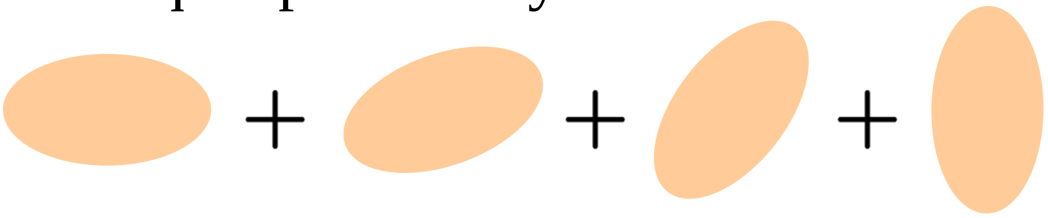
$$(I = \mathcal{J}\omega, \ \ \omega = \dot{\theta})$$

^{154}Sm is deformed

(note) What is 0^+ state (Quantum Mechanics)?

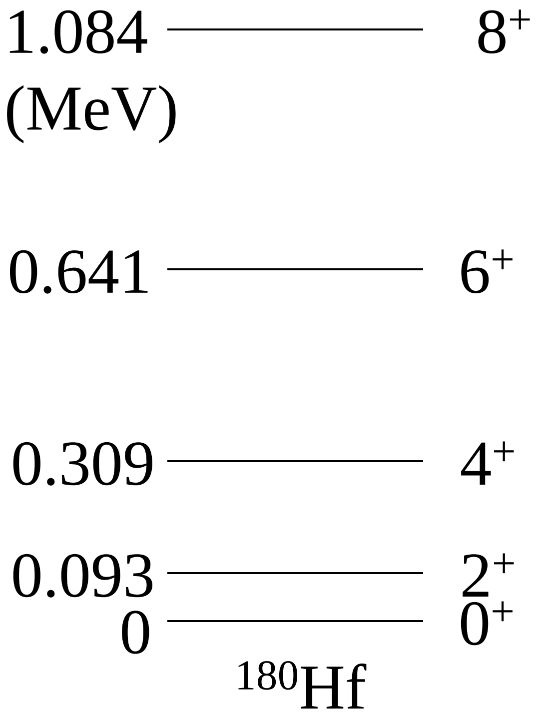
0^+: no preference of direction (spherical)

➡ Mixing of all orientations with an equal probability



c.f. HF + Angular Momentum Projection

Evidences for nuclear deformation



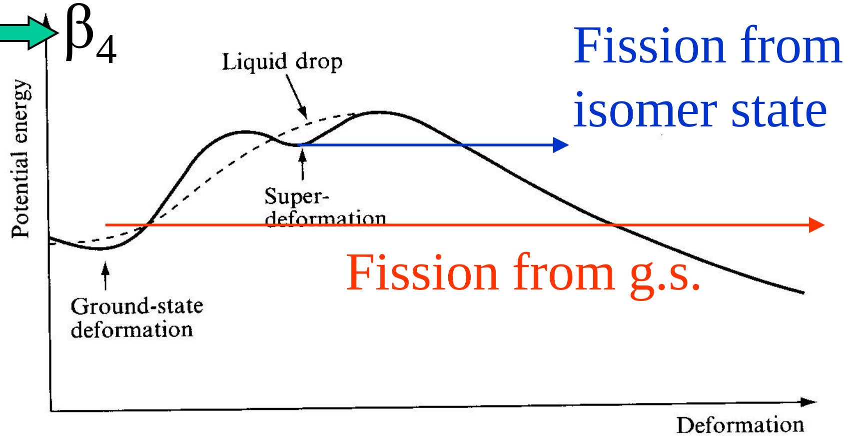
- The existence of rotational bands

$$E_I = \frac{I(I+1)\hbar^2}{2\mathcal{J}}$$

- Very large quadrupole moments (for odd-A nuclei)

$$Q = e\sqrt{\frac{16\pi}{5}}\langle\Psi_{II}|r^2Y_{20}|\Psi_{II}\rangle$$

- Strongly enhanced quadrupole transition probabilities
- Hexadecapole matrix elements ⟷ β_4
- Single-particle structure
- Fission isomers



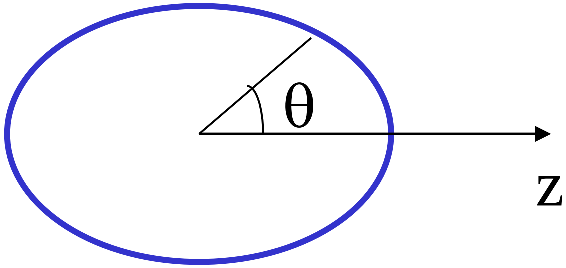
Single-Particle Motion in a Deformed Potential

$$V(\boldsymbol{r}) \sim \int v(\boldsymbol{r},\boldsymbol{r}')\rho(\boldsymbol{r}')d\boldsymbol{r}' \sim -g\rho(\boldsymbol{r}) \qquad \text{if} \quad v(\boldsymbol{r},\boldsymbol{r}') = -g\delta(\boldsymbol{r}-\boldsymbol{r}')$$

If the density is deformed, the mean-field potential is also deformed

(note)
for an axially symmetric spheroid: $R(\theta) = R_0(1+\beta_2 Y_{20}(\theta))$

Change R_0 by $R(\theta)$
$V(r) = -V_0/[1+\exp((r-R_0)/a)]$
in a Woods-Saxon potential



Deformed Woods-Saxon potential

$$\begin{aligned} V(r,\theta) &= -V_0/[1+\exp((r-R_0-R_0\beta_2 Y_{20}(\theta))/a] \\ &\sim V_0(r) - \beta_2 R_0 \frac{dV_0}{dr} Y_{20}(\theta) + \cdots \end{aligned}$$

Single-particle motion in a deformed potential

Deformed Woods-Saxon potential

$$\boxed{\begin{aligned} V(r,\theta) &= -V_0/[1+\exp((r-R_0-R_0\beta_2 Y_{20}(\theta))/a] \\ &\sim V_0(r)-\beta_2 R_0\frac{dV_0}{dr}Y_{20}(\theta)+\cdots \end{aligned}}$$

A potential without rotational symmetry

→ Angular momentum is not a good quantum number
(it does not conserve)

■Let us investigate the effect of the Y_{20} term using the first order perturbation theory

(note) first order perturbation theory

$$H = H_0 + H_1$$

Suppose we know the eigenvalues and the eigenfunctions of H_0 :

$$H_0|\phi_n^{(0)}\rangle = E_n^{(0)}|\phi_n^{(0)}\rangle$$

H_1 causes the change of the eigenvalues and the eigenfunctions as:

$$\begin{aligned} E_n &= E_n^{(0)} + \langle\phi_n^{(0)}|H_1|\phi_n^{(0)}\rangle + \cdots \\ |\phi_n\rangle &= |\phi_n^{(0)}\rangle + \sum_{m\neq n} \frac{\langle\phi_m^{(0)}|H_1|\phi_n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}} |\phi_m\rangle + \cdots \end{aligned}$$

Single-particle motion in a deformed potential

Deformed Woods-Saxon potential

$$\boxed{V(r,\theta) \sim V_0(r) - \beta_2 R_0 \frac{dV_0}{dr} Y_{20}(\theta) + \cdots}$$

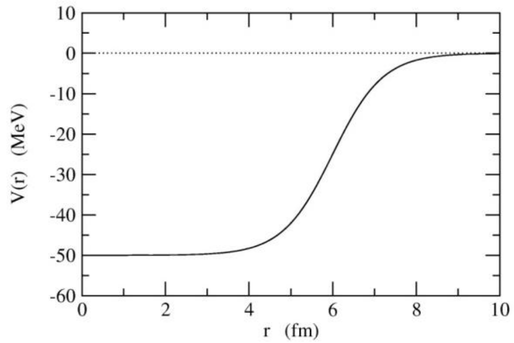
■Let us investigate the effect of the Y_{20} term with the perturbation theory:

Eigenfunctions for β=0 (spherical): $\psi_{nlK}(\boldsymbol{r}) = R_{nl}(r) Y_{lK}(\hat{\boldsymbol{r}})$

Eigenvalues: E_{nl} (independent of K)

The energy change:

$$\begin{aligned} E_{nl} &\rightarrow E_{nl} + \langle \psi_{nlK} | \Delta V | \psi_{nlK} \rangle \\ &= E_{nl} - \beta_2 R_0 \underbrace{\left[\int_0^\infty r^2 dr\, \frac{dV_0}{dr} (R_{nl}(r))^2\right]}_{\text{positive}} \cdot \underbrace{\langle Y_{lK} | Y_{20} | Y_{lK} \rangle}_{\propto\, -(3K^2 - l(l+1))} \end{aligned}$$



Single-particle motion in a deformed potential

Deformed Woods-Saxon potential

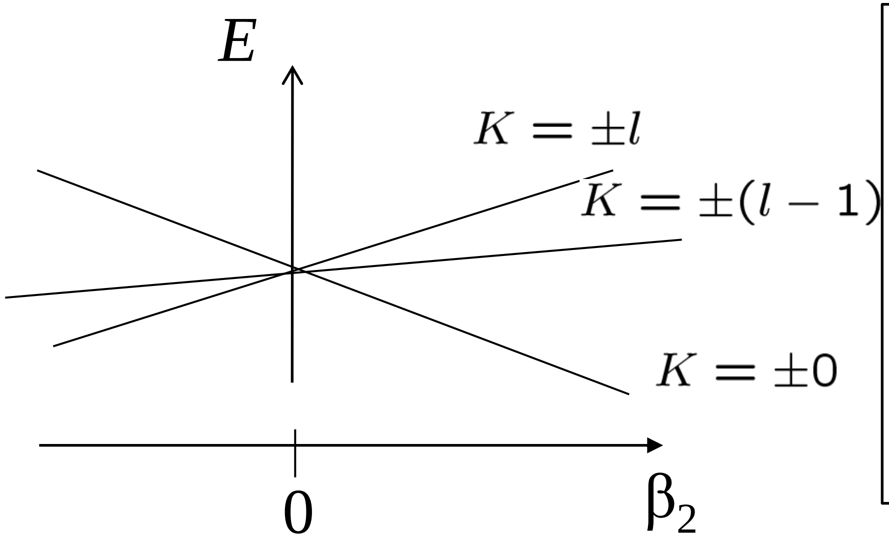
$$\psi_{nlK}(\boldsymbol{r}) = R_{nl}(r)Y_{lK}(\hat{\boldsymbol{r}})$$

$$\boxed{V(r,\theta) \sim V_0(r) - \beta_2 R_0 \frac{dV_0}{dr} Y_{20}(\theta) + \cdots}$$

■Let us investigate the effect of the Y_{20} term with the perturbation theory:

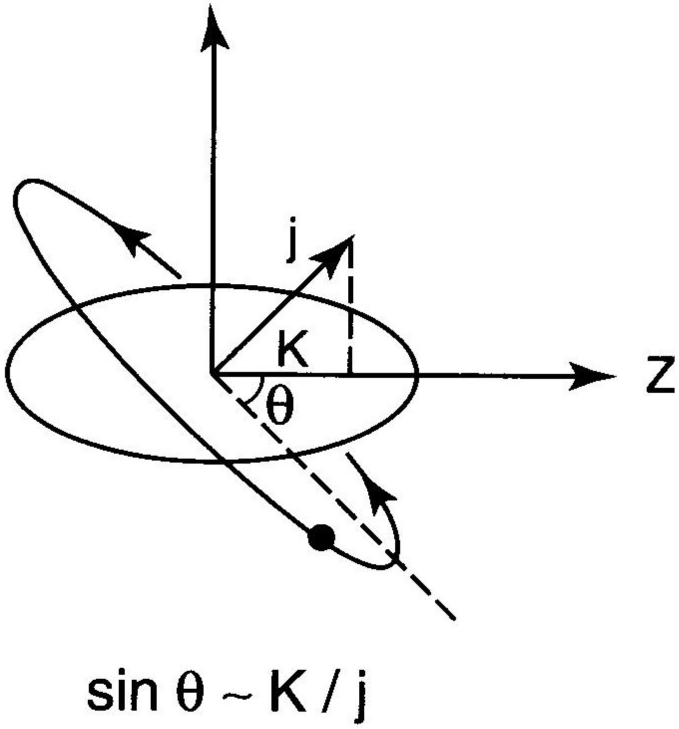
The energy change:

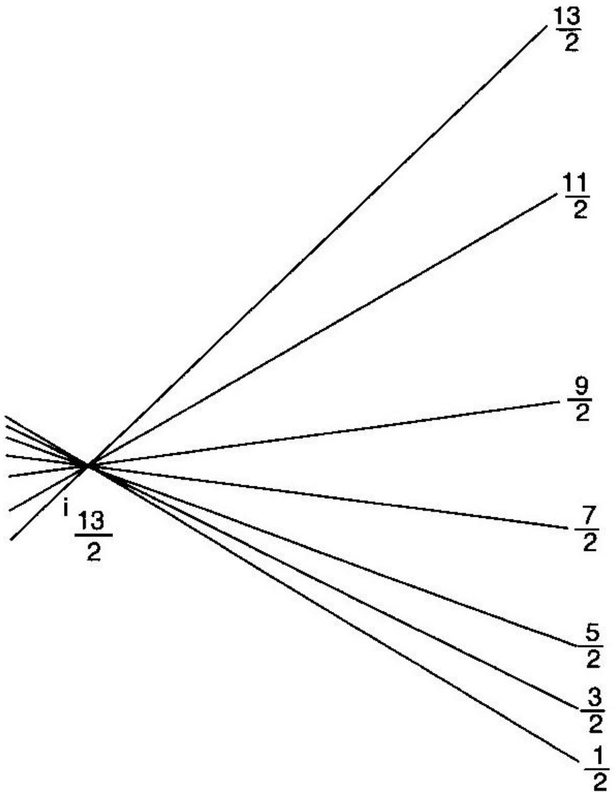
$$E_{nl} \quad \to \quad E_{nl} + \alpha_{nl}\,\beta_2\,(3K^2 - l(l+1)) \qquad\qquad (\alpha_{nl} > 0)$$



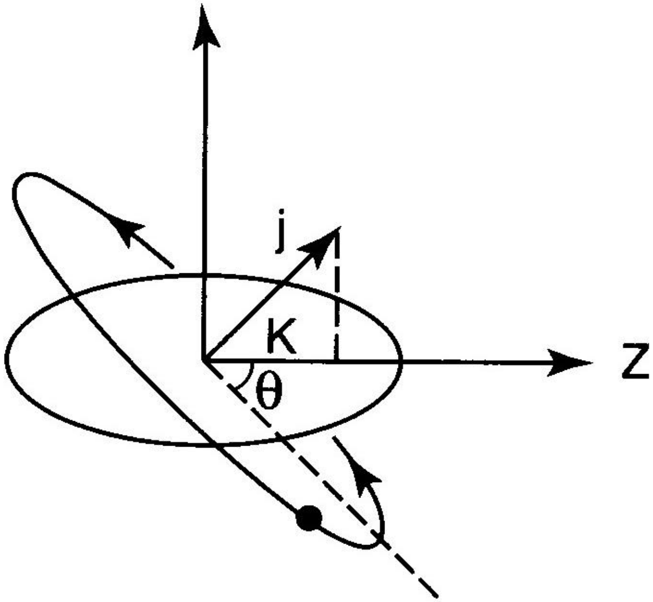
- The energy change depends on *K* (the degeneracy is broken)
- if $\beta_2 > 0$, the lower the energy is for the smaller K
- opposite if $\beta_2 < 0$
- degeneracy between *K* and –*K*

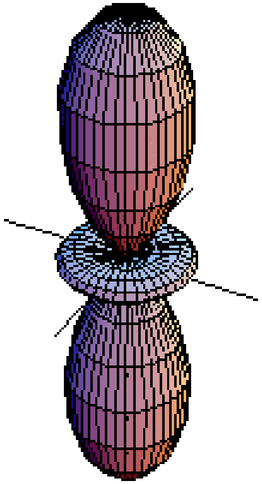
Geometrical interpretation



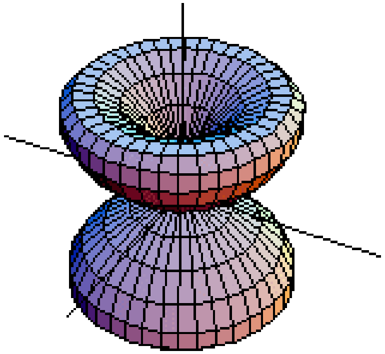


- *K* is the projection of angular momentu on the z-axis
- nucleon moves in a plane perpendicular to the ang. mom. vector
- for prolate deformation, a nucleon with small K moves along the z-axis
- therefore, it feels more attraction and the energy is lowered
- for large K, the nucleon moves along the x-axis and feels less attraction

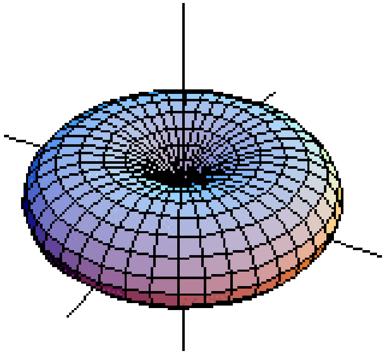




r = Y_{20}
(K=0)



r = Y_{21}
(K=1)



r = Y_{22}
(K=2)

Single-particle motion in a deformed potential

$$\boxed{V(r,\theta) \sim V_0(r) - \beta_2 R_0 \frac{dV_0}{dr} Y_{20}(\theta) + \cdots}$$

■Let us investigate the effect of the Y_{20} term with the perturbation theory:

Now the wave function: $$|\phi_n\rangle \;=\; |\phi_n^{(0)}\rangle + \sum_{m\neq n} \frac{\langle \phi_m^{(0)}|H_1|\phi_n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}}|\phi_m\rangle + \cdots$$

The eigenfunctions for β=0 (spherical): $$\psi_{nlK}(\boldsymbol{r}) = R_{nl}(r)Y_{lK}(\hat{\boldsymbol{r}})$$

$$\psi_{nlK} \to \psi_{nlK} + \sum_{n'l'K'} \frac{\langle \psi_{n'l'K'}|\Delta V|\psi_{nlK}\rangle}{E_{nl} - E_{n'l'}} \psi_{n'l'K'}$$

mixing of the states connected by $\langle Y_{l'K'}|Y_{20}|Y_{lK}\rangle$

- *l* is not conserved
- For axially symmetric deformation (Y_{20}), *K* does not change (*K'* = *K*), i.e., *K* is conserved
- Y_{20} does not change the parity, thus paritiy is also conserved.

Single-particle motion in a deformed potential

$$\boxed{V(r,\theta) \sim V_0(r) - \beta_2 R_0 \frac{dV_0}{dr} Y_{20}(\theta) + \cdots}$$

In general

$$\Psi_K(\boldsymbol{r}) = \sum_l \frac{u_{lK}(r)}{r} Y_{lK}(\hat{\boldsymbol{r}})$$

* u_{lK} could be expanded on the eigen-functions of a spherical potential:

$$u_{lK}(r) = \sum_n \alpha_{nlK}\, u_{nl}(r)$$

examples)

$$\begin{aligned}
|K^\pi\rangle = \left|0^+\right\rangle &= A_s\,|Y_{00}\rangle + A_d\,|Y_{20}\rangle + A_g\,|Y_{40}\rangle + \cdots \\
\left|1^+\right\rangle &= B_d\,|Y_{21}\rangle + B_g\,|Y_{41}\rangle + \cdots \\
\left|0^-\right\rangle &= C_p\,|Y_{10}\rangle + C_f\,|Y_{30}\rangle + C_h\,|Y_{50}\rangle + \cdots
\end{aligned}$$

Nilsson Hamiltonian

$$\begin{aligned} V_{\mathrm{Nil}} &= \frac{1}{2}m\omega_\perp^2(x^2+y^2)+\frac{1}{2}m\omega_z^2z^2 \\ &\quad + C\boldsymbol{l}\cdot\boldsymbol{s}+D(\boldsymbol{l}^2-\langle\boldsymbol{l}^2\rangle_N) \end{aligned}$$

(Anisotropic H.O. + correction + spin-orbit)

$$\begin{aligned} \omega_\perp^2 &= \omega_0^2(1+2\delta/3) \\ \omega_z^2 &= \omega_0^2(1-4\delta/3) \end{aligned}$$

(note) $\omega_x\,\omega_y\,\omega_z=\omega_0^3=\mathrm{const.}$

$$\begin{aligned} &\frac{1}{2}m\omega_\perp^2(x^2+y^2)+\frac{1}{2}m\omega_z^2z^2 \\ &\qquad = \frac{1}{2}m\omega_0^2r^2-m\omega_0^2\beta\,r^2Y_{20}(\theta) \end{aligned}$$

$$\beta=\frac{\delta}{3}\sqrt{\frac{16\pi}{5}}$$

Nilsson Diagram

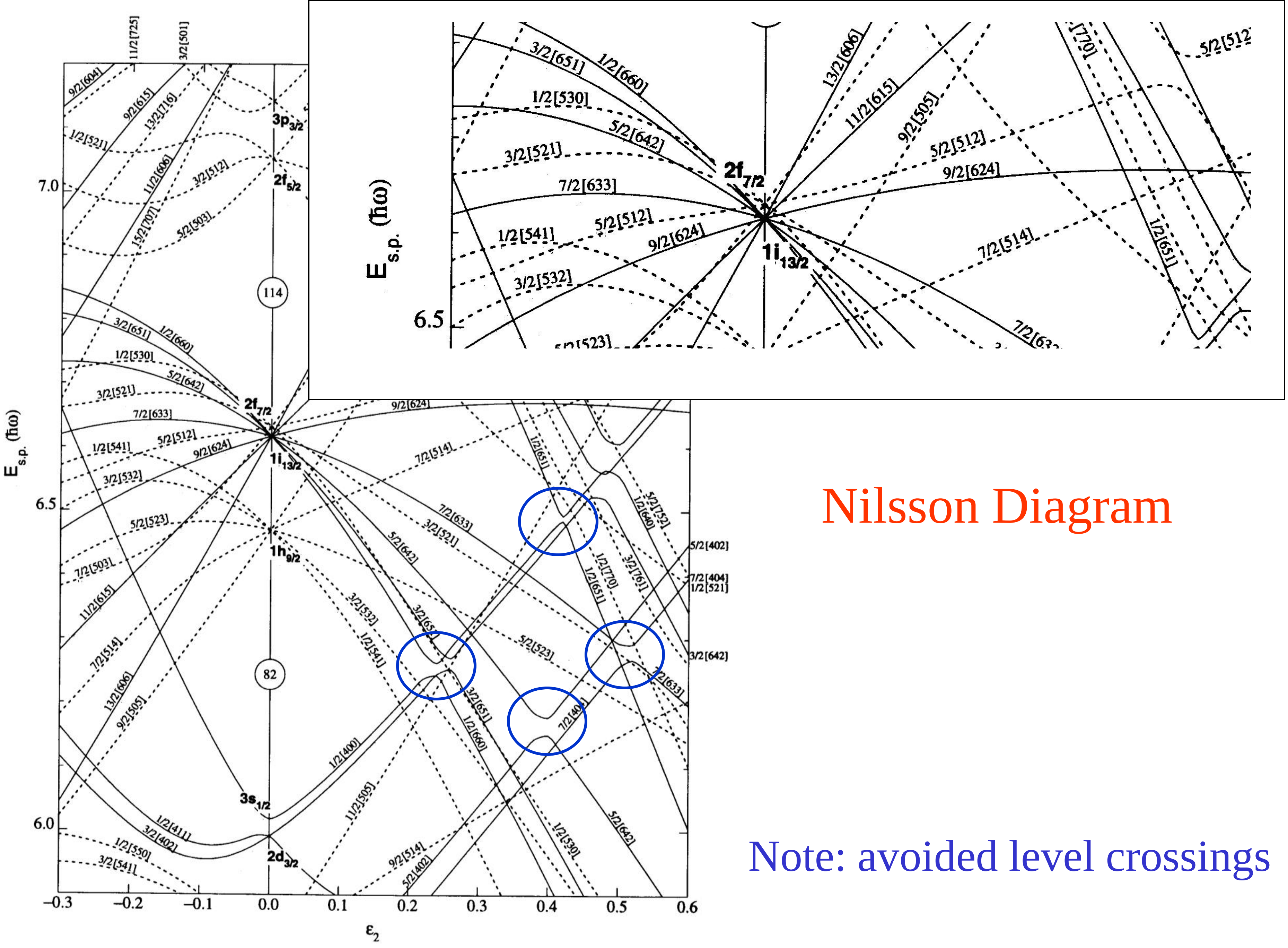
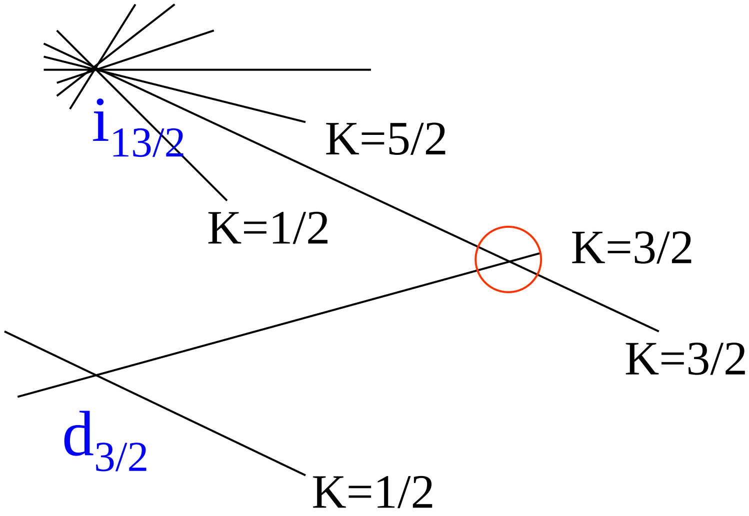


Figure 13. Nilsson diagram for protons, Z ≥ 82 ($\epsilon_4 = \epsilon_2^2/6$).

Note: avoided level crossings

Avoided level crossing

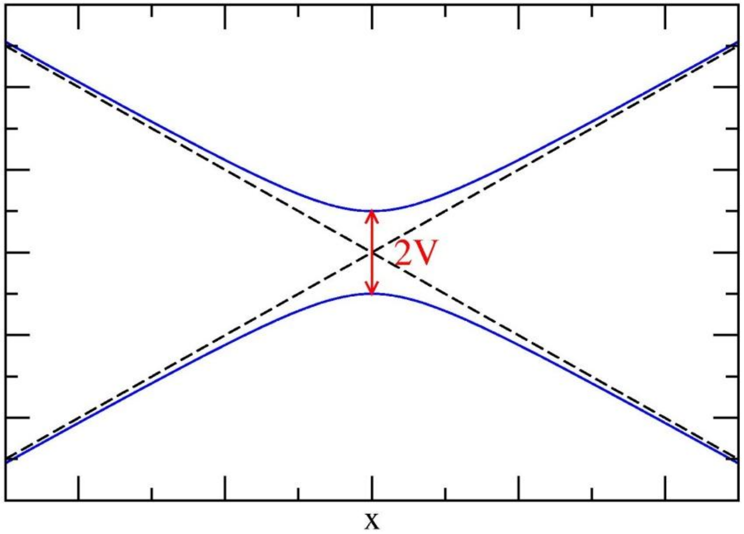


Example:

$$\begin{pmatrix} -\epsilon x & V \\ V & \epsilon x \end{pmatrix}$$

$$\rightarrow \lambda_\pm(x) = \pm\sqrt{\epsilon^2 x^2 + V^2}$$

diagonalization



Interaction between $|\mathcal{Y}_{\frac{13}{2},6,\frac{3}{2}}\rangle$ and $|\mathcal{Y}_{\frac{3}{2},2,\frac{3}{2}}\rangle$

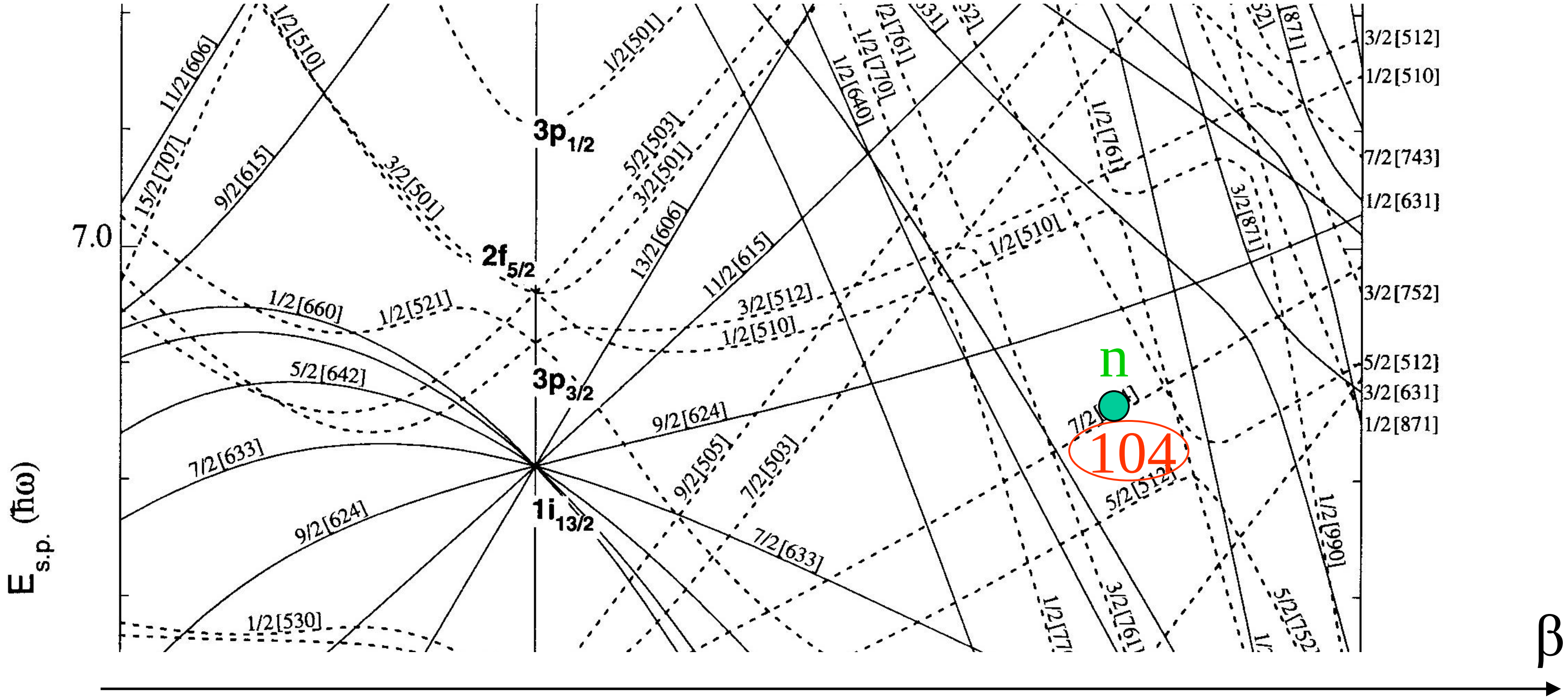
$$\begin{pmatrix} \langle\mathcal{Y}_{\frac{13}{2},6,\frac{3}{2}}|H|\mathcal{Y}_{\frac{13}{2},6,\frac{3}{2}}\rangle & \langle\mathcal{Y}_{\frac{13}{2},6,\frac{3}{2}}|H|\mathcal{Y}_{\frac{3}{2},2,\frac{3}{2}}\rangle \\ \langle\mathcal{Y}_{\frac{3}{2},2,\frac{3}{2}}|H|\mathcal{Y}_{\frac{13}{2},6,\frac{3}{2}}\rangle & \langle\mathcal{Y}_{\frac{3}{2},2,\frac{3}{2}}|H|\mathcal{Y}_{\frac{3}{2},2,\frac{3}{2}}\rangle \end{pmatrix}$$

Two levels with the same quantum numbers never cross
(an infinitesimal interaction causes them to repel).

“avoided crossing” or “level repulsion”

Single-particle spectra of deformed odd-A nuclei

Nilsson diagram: each level has two-fold degeneracy ($\pm K$)



5/2$^-$ 0.508

9/2$^+$ 0.321

7/2$^-$

$^{177}_{72}$Hf$_{105}$