Random Phase Approximation

Tamm-Dancoff Approximation: $|\nu\rangle = Q_\nu^\dagger|HF\rangle = \sum_{ph} X_{ph}\, a_p^\dagger a_h|HF\rangle$

(superposition of 1p1h states)

$$[H, Q_\nu^\dagger] \approx E_\nu Q_\nu^\dagger$$

$$\Longleftrightarrow \quad \langle HF|[\delta Q, [H, Q_\nu^\dagger]]|HF\rangle = E_\nu \langle HF|[\delta Q, Q_\nu^\dagger]|HF\rangle$$

Drawbacks:

- No influence of v in the ground state

$$E_{coll} = \epsilon + \lambda \sum_{ph} |D_{ph}|^2$$ ⇐ Interaction is essential in describing collective excitations

- Energy Weighted Sum Rule is violated in TDA
- Admixture of the spurious modes with the physical excitation modes

HF ⟺ Broken Symmetries (CM localization, rotation,…….)

Restoration of broken symmetries ⇒ Goldstone mode (spurious motion)

A better approximation: **the random phase approximation (RPA)**

$$\boxed{|\nu\rangle = Q_\nu^\dagger|0\rangle = \sum_{ph}\left(X_{ph}\,a_p^\dagger a_h - Y_{ph}\,a_h^\dagger a_p\right)|0\rangle}$$

(superposition of 1p1h states)

$$[H, Q_\nu^\dagger] \approx E_\nu Q_\nu^\dagger$$

$$\Longleftrightarrow \quad \langle 0|[\delta Q, [H, Q_\nu^\dagger]]|0\rangle = E_\nu \langle 0|[\delta Q, Q_\nu^\dagger]|0\rangle$$

δQ : arbitrary operator

(note) Harmonic oscillator: $H_{\rm HO} = \hbar\omega(a^\dagger a + 1/2)$

$$\Longrightarrow \quad [H_{\rm HO}, a^\dagger] = \hbar\omega a^\dagger$$

↷ RPA: describes a harmonic motion

- RPA ground state: $\boxed{Q_\nu|0\rangle = 0}$
- Normalization: $\langle\nu|\nu\rangle = 1 \rightarrow \boxed{\sum_{ph}|X_{ph}|^2 - |Y_{ph}|^2 = 1}$

$$Q_\nu^\dagger = \sum_{ph} X_{ph}\, a_p^\dagger a_h - Y_{ph}\, a_h^\dagger a_p$$

$$\langle 0 | [\delta Q, [H, Q_\nu^\dagger]] | 0 \rangle = E_\nu \langle 0 | [\delta Q, Q_\nu^\dagger] | 0 \rangle$$

$$\begin{cases} H = \sum_{1,2} t_{12}\, a_1^\dagger\, a_2 + \frac{1}{4} \sum_{1,2,3,4} \bar{v}_{1234}\, a_1^\dagger\, a_2^\dagger\, a_4\, a_3 \\ \delta Q = a_h^\dagger a_p, \qquad a_p^\dagger a_h \end{cases}$$

➡ Equations for X_{ph} and Y_{ph}

Quasi-boson approximation:

$$\langle 0 | [\delta Q, [H, Q_\nu^\dagger]] | 0 \rangle \approx \langle HF | [\delta Q, [H, Q_\nu^\dagger]] | HF \rangle$$

$$\langle 0 | [\delta Q, Q_\nu^\dagger] | 0 \rangle \approx \langle HF | [\delta Q, Q_\nu^\dagger] | HF \rangle$$

(note)

$$\langle 0 | [Q_\nu, Q_\nu^\dagger] | 0 \rangle \approx \langle HF | [Q_\nu, Q_\nu^\dagger] | HF \rangle = 1$$

$$\langle HF|[\delta Q,[H,Q_\nu^\dagger]]|HF\rangle = E_\nu \langle HF|[\delta Q, Q_\nu^\dagger]|HF\rangle$$

$$Q_\nu^\dagger = \sum_{ph} X_{ph}\, a_p^\dagger a_h - Y_{ph}\, a_h^\dagger a_p \qquad \delta Q = a_h^\dagger a_p, \quad a_p^\dagger a_h$$

RPA equation:

$$\boxed{\begin{aligned} \sum_{p'h'} A_{ph,p'h'} X_{p'h'} + B_{ph,p'h'} Y_{p'h'} &= E_\nu X_{ph} \\ \sum_{p'h'} B^*_{ph,p'h'} X_{p'h'} + A^*_{ph,p'h'} Y_{p'h'} &= -E_\nu Y_{ph} \end{aligned}}$$

$$\begin{aligned} A_{ph,p'h'} &= (\epsilon_p - \epsilon_h)\delta_{ph,p'h'} + \langle ph'|\bar{v}|hp'\rangle \\ B_{ph,p'h'} &= \langle pp'|\bar{v}|hh'\rangle \end{aligned}$$

or

$$\boxed{\begin{pmatrix} A & B \\ -B^* & -A^* \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = E_\nu \begin{pmatrix} X \\ Y \end{pmatrix}}$$

Random Phase Approximation: Historical note

D. Bohm and D. Pines, Phys. Rev. 92('53)609

The plasma oscillation in an infinite electron gas

$$H = \frac{1}{2m}\sum_i \left(\boldsymbol{p}_i + \frac{e}{c}\boldsymbol{A}(\boldsymbol{x}_i)\right)^2 + \frac{1}{8\pi}\int \boldsymbol{E}(\boldsymbol{x})^2 d\boldsymbol{x} - 2\pi n e^2 \sum_k \frac{1}{k^2}$$

$$\boldsymbol{A}(\boldsymbol{x}) = \sqrt{4\pi c^2}\sum_k q_k \boldsymbol{\epsilon}_k e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\boxed{\begin{aligned} \sum_i \boldsymbol{A}(\boldsymbol{x}_i)^2 \quad &= \quad \sum_{ikl} \boldsymbol{\epsilon}_k \cdot \boldsymbol{\epsilon}_l \, q_k q_l \, e^{i(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{x}_i} \\ &\rightarrow \quad \sum_{ik} q_k q_{-k} \qquad (\boldsymbol{k}+\boldsymbol{l}=0 \ \text{ only}) \end{aligned}}$$

RPA on a schematic model

Separable interaction: $\langle ph'|\bar{v}|hp'\rangle = \lambda D_{ph} D^*_{p'h'}$

$$\boxed{\frac{1}{\lambda} = \sum_{ph} \frac{|D_{ph}|^2}{E-\epsilon_{ph}} - \frac{|D_{ph}|^2}{E+\epsilon_{ph}}}$$

(RPA dispersion relation)

Cf. TDA dispersion relation: $\frac{1}{\lambda} = \sum_{ph} \frac{|D_{ph}|^2}{E-\epsilon_{ph}}$

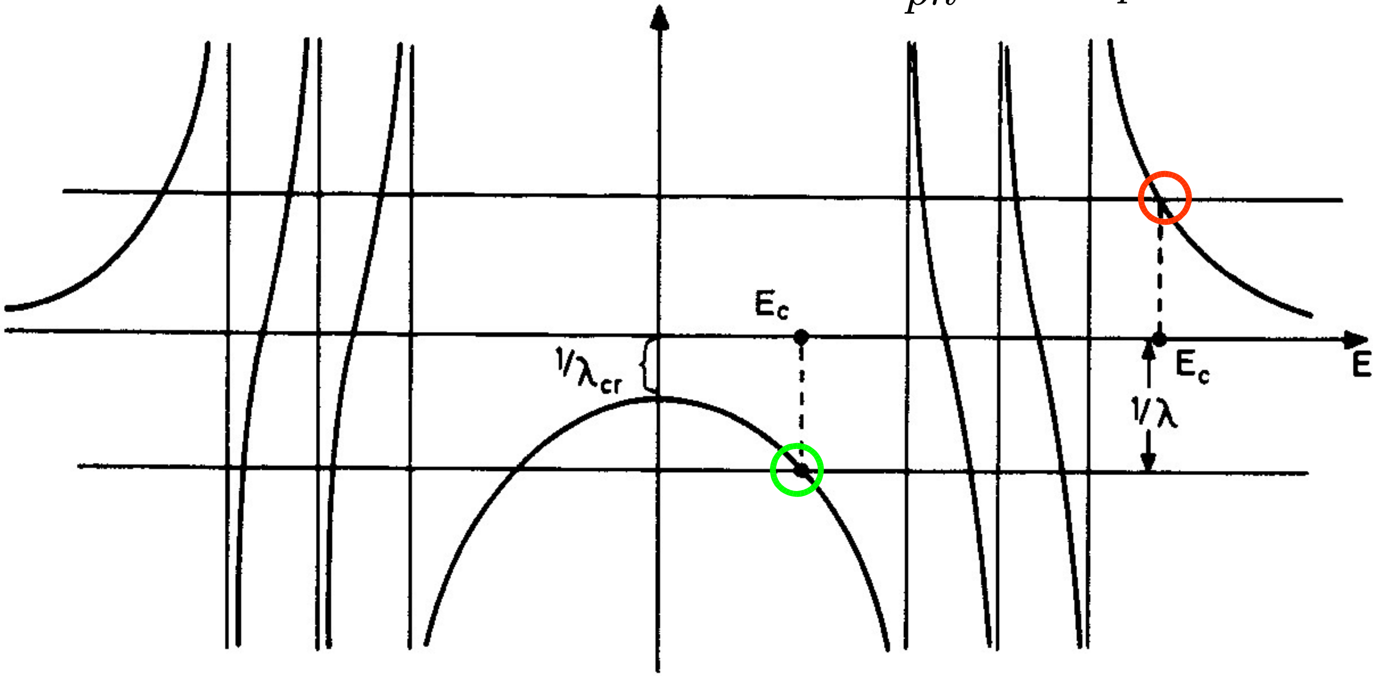
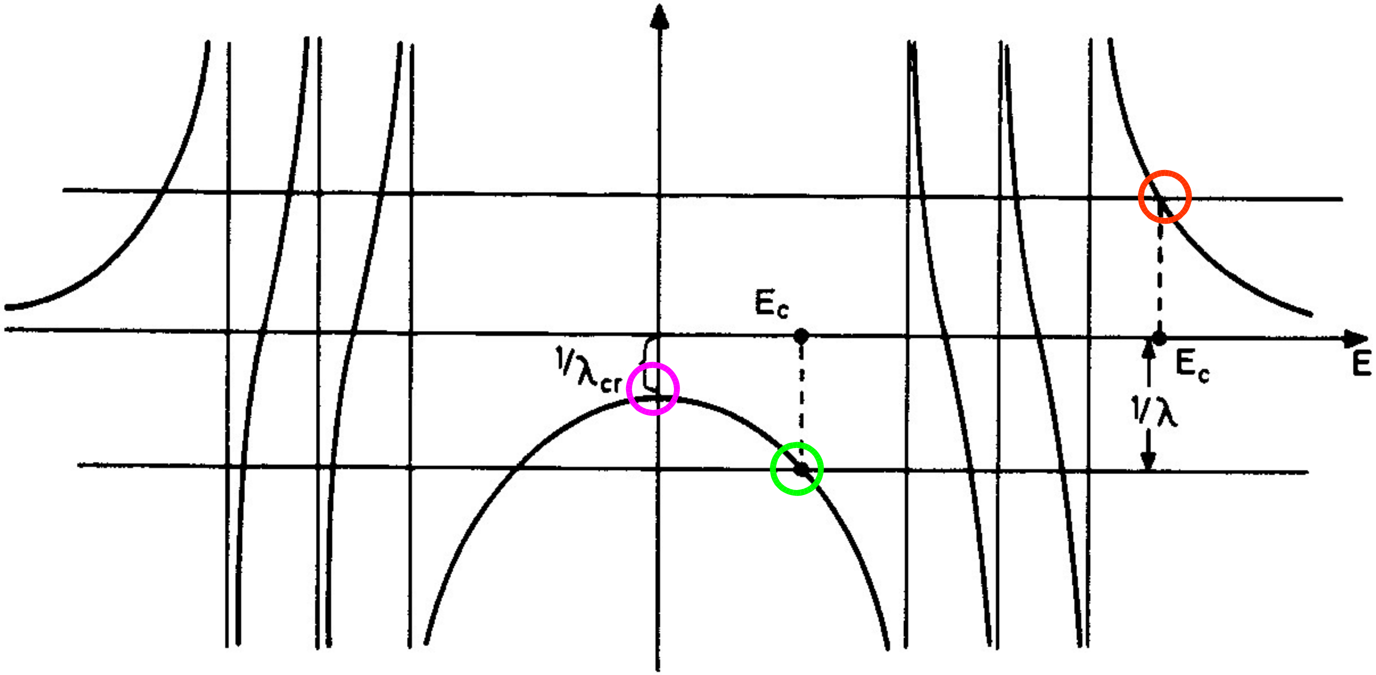


Figure 8.11. Graphical solution of the dispersion relation (8.135).

$$\boxed{\frac{1}{\lambda} = \sum_{ph} \frac{|D_{ph}|^2}{E - \epsilon_{ph}} - \frac{|D_{ph}|^2}{E + \epsilon_{ph}}}$$

(RPA dispersion relation)



i) Critical strength for attractive interaction

$$\lambda > \lambda_{crit} \rightarrow E^2 < 0 \Longleftrightarrow \text{Instability of the HF state}$$

ii) Symmetric between *E* and –*E*

iii) In the degenerate limit $E^2 = \epsilon^2 + 2\epsilon\lambda\sum_{ph}|D_{ph}|^2$

Spurious motion in RPA

Mean-Field Approximation ⟷ Broken symmetiries

- Center of mass localization (single center)
- Rotational motion

Restoration of broken symmetries

⇒ Zero mode (Nambu-Goldstone mode)

RPA $\langle HF|[\delta Q,[H,Q_\nu^\dagger]]|HF\rangle = E_\nu \langle HF|[\delta Q,Q_\nu^\dagger]|HF\rangle$

if $[H,\hat{O}]=0$

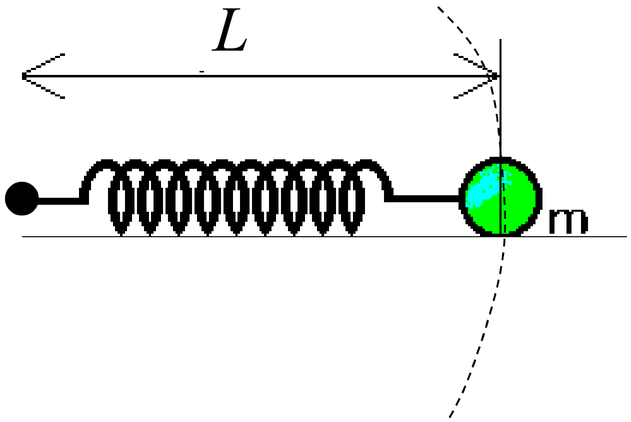
Then $\hat{O}$ is a solution of RPA with $E=0$

$$\hat{O}=\sum_{ph}(O_{ph}a_p^\dagger a_h + O_{hp}a_h^\dagger a_p)$$

The physical solutions are exactly separated out from the spurious modes.

(note) rigid rotation of mechanical systems

E.R. Marshalek, Ann. of Phys. 53('69) 569



$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}k(\sqrt{x^2+y^2} - L)^2$$

Random phase approximation:

- Small oscillation around equillibrium

$$V(x,y) \sim V(x_0,y_0) + \frac{1}{2}\sum_{i,j}(\partial_i\partial_j V)(x_i - x_{i0})(x_j - x_{j0})$$

- All degrees of freedom are treated equally

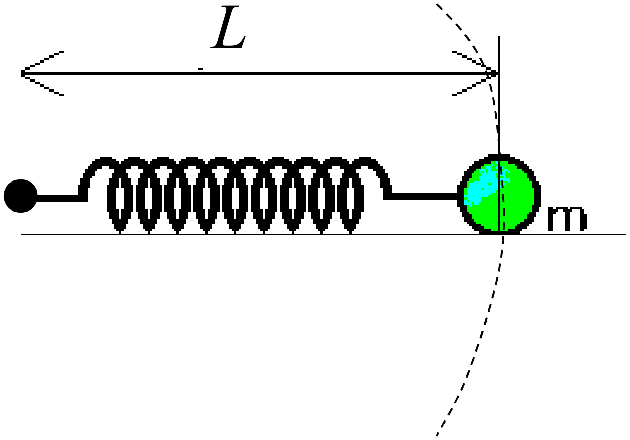
⇨ Treat *x* and *y* on the same footing
(work with the Cartesian coordinate)

i) “Spherical” case (L=0)

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}k(x^2 + y^2)$$

$$\Longrightarrow \omega_x = \omega_y = \sqrt{k/m}$$

ii) “Deformed” case ($L \neq 0$)



$$x_0 = L, \;\; y_0 = 0$$

⟸ Spontaneous Symm. Breaking

(note)
$$\begin{aligned} \frac{\partial^2}{\partial x^2}(r-L)^2 &= \frac{2}{r^3}(r^3 - Ly^2) \\ \frac{\partial^2}{\partial y^2}(r-L)^2 &= \frac{2}{r^3}(r^3 - Lx^2) \end{aligned}$$

↷

$$H \sim \frac{p_x^2}{2m} + \frac{1}{2}kx^2 + \frac{p_y^2}{2m}$$

↷ $\omega_x = \sqrt{k/m}, \;\; \underline{\underline{\omega_y = 0}}$

RPA correlation energy

$$\begin{pmatrix} A & B \\ -B^* & -A^* \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = E \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\begin{aligned} A_{ph,p'h'} &= \langle HF|[a_h^\dagger a_p, [H, a_{p'}^\dagger a_{h'}]]|HF\rangle \\ B_{ph,p'h'} &= -\langle HF|[a_p^\dagger a_h, [H, a_{h'}^\dagger a_{p'}]]|HF\rangle \end{aligned}$$

$$a_p^\dagger a_h \rightarrow \hat{B}_{ph}^\dagger, \quad a_h^\dagger a_p \rightarrow \hat{B}_{ph}$$

$$\begin{aligned} H &= E_{HF} + \sum_{php'h'} A_{ph,p'h'} \hat{B}_{ph}^\dagger \hat{B}_{p'h'} \\ &\quad + \frac{1}{2} \sum_{php'h'} (B_{ph,p'h'} \hat{B}_{ph}^\dagger \hat{B}_{p'h'}^\dagger + h.c.) \end{aligned}$$

Bogoliubov transformation: $O_\nu^\dagger = \sum_{ph} X_{ph} \hat{B}_{ph}^\dagger - Y_{ph} \hat{B}_{ph}$

$$H = E_{HF} + E_{corr} + \sum_\nu E_\nu O_\nu^\dagger O_\nu$$

$$\boxed{E_{corr} = \frac{1}{2}\left(\sum_\nu E_\nu - Tr(A)\right)}$$

RPA correlation energy

Three-level Lipkin model

K. Hagino and G.F. Bertsch, PRC61('00)024307

$$H = \epsilon(\hat{n}_1 + \hat{n}_2) - \frac{V}{2}(K_1K_1 + K_2K_2 + h.c.)$$

$$\hat{n}_\alpha = \sum_{i=1}^{N} c^\dagger_{\alpha i}c_{\alpha i}, \quad K_\alpha = \sum_{i=1}^{N} c^\dagger_{\alpha i}c_{0i} \qquad \alpha = 0, 1, 2$$