▨ 角運動量の合成（スピン－軌道力）

◦ 複習

$$[l_i, l_j] = i\varepsilon_{ijk} l_k$$

$$[l_x, l_y] = i l_z$$
$$[l_y, l_z] = i l_x$$
$$[l_z, l_x] = i l_y$$

$$l_\pm \equiv l_x \pm i l_y$$

$$l_\pm |lm\rangle = \sqrt{(l \mp m)(l \pm m + 1)}\ |l\, m \pm 1\rangle$$

◦ l-s 力

$$\vec{j} = \vec{l} + \vec{s}$$

(note) $$\vec{j}^{\,2} = \vec{l}^{\,2} + \vec{s}^{\,2} + 2\vec{l}\cdot\vec{s}$$
$$= \vec{l}^{\,2} + \vec{s}^{\,2} + 2 l_z s_z + l_+ s_- + l_- s_+$$

角度成分とスピン部分の積 ： $Y_{lm}\, \chi_{m_s}$

$m_j = m_l + m_s$		
$l+\frac{1}{2}$	$Y_{ll}\,\chi_\uparrow$	
$l-\frac{1}{2}$	$Y_{l\,l-1}\,\chi_\uparrow$	$Y_{ll}\,\chi_\downarrow$
$l-\frac{3}{2}$	$Y_{l\,l-2}\,\chi_\uparrow$	$Y_{l\,l-1}\,\chi_\downarrow$
$\vdots$		
$-l-\frac{1}{2}$	$Y_{l\,-l}\,\chi_\downarrow$	

$$\vec{j}^{\,2}\, Y_{ll}\, \chi_\uparrow = \left(\vec{l}^{\,2} + \vec{s}^{\,2} + 2 l_z s_z + \cancel{l_+ s_-} + \cancel{l_- s_+}\right) Y_{ll}\, \chi_\uparrow$$

$$= \Big[\underbrace{l(l+1) + \tfrac{3}{4} + l}_{\substack{\shortparallel \\ l^2 + 2l + \frac{3}{4} \,=\, (l+\frac{1}{2})(l+\frac{3}{2})}}\Big]\, Y_{ll}\, \chi_\uparrow$$

∴ $\quad Y_{ll}\, \chi_\uparrow = |l+\tfrac{1}{2},\ l+\tfrac{1}{2}\rangle$

$$(l_- + s_-)\, Y_{ll}\, \chi_\uparrow = j_-\, |l+\tfrac{1}{2},\ l+\tfrac{1}{2}\rangle$$

$$= \sqrt{(l+\tfrac{1}{2} + l+\tfrac{1}{2})(l+\tfrac{1}{2} - l-\tfrac{1}{2} + 1)} \times |l+\tfrac{1}{2},\ l-\tfrac{1}{2}\rangle$$

$$= \sqrt{2l+1}\ |l+\tfrac{1}{2},\ l-\tfrac{1}{2}\rangle$$

$$= \sqrt{2l}\ Y_{l\,l-1}\, \chi_\uparrow + Y_{ll}\, \chi_\downarrow$$

∴ $\boxed{|l+\tfrac{1}{2},\ l-\tfrac{1}{2}\rangle = \sqrt{\dfrac{2l}{2l+1}}\ Y_{l\,l-1}\, \chi_\uparrow + \dfrac{1}{\sqrt{2l+1}}\ Y_{ll}\, \chi_\downarrow}$

これと直交する状態を作ってみると：

$$\frac{1}{\sqrt{2l+1}}\ Y_{l\,l-1}\, \chi_\uparrow - \sqrt{\frac{2l}{2l+1}}\ Y_{ll}\, \chi_\downarrow$$

(note) $\vec{j}^{\,2} \left(Y_{l\,l-1}\, \chi_\uparrow - \sqrt{2l}\ Y_{ll}\, \chi_\downarrow\right)$

$$= \left(\vec{l}^{\,2} + \vec{s}^{\,2} + 2 l_z s_z + l_+ s_- + l_- s_+\right)\left(Y_{l\,l-1}\, \chi_\uparrow - \sqrt{2l}\ Y_{ll}\, \chi_\downarrow\right)$$

$$= \left(l(l+1) + \frac{3}{4} + 2(l-1)\cdot\frac{1}{2} \right) \times Y_{l\,l-1}\chi_\uparrow + \sqrt{(l-l+1)(l+l-1+1)}\,\sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)} \times Y_{ll}\chi_\downarrow$$

$$- \sqrt{2l}\left(l(l+1) + \frac{3}{4} + 2l\cdot\left(-\frac{1}{2}\right) \right) Y_{ll}\chi_\downarrow - \sqrt{2l}\,\sqrt{2l}\,Y_{l\,l-1}\chi_\uparrow$$

$$= \left(l^2 + l + \frac{3}{4} + l - 1 - 2l \right) Y_{l\,l-1}\chi_\uparrow$$

$$- \sqrt{2l}\left(l^2 + l - l + \frac{3}{4} - 1 \right) Y_{ll}\chi_\downarrow$$

$$= \left(l^2 - \frac{1}{4} \right)\left(Y_{l\,l-1}\chi_\uparrow - \sqrt{2l}\,Y_{ll}\chi_\downarrow \right)$$

$\Updownarrow$

$$\left| l-\frac{1}{2},\, l-\frac{1}{2} \right\rangle = \frac{1}{\sqrt{2l+1}}\,Y_{l\,l-1}\chi_\uparrow - \sqrt{\frac{2l}{2l+1}}\,Y_{ll}\chi_\downarrow$$

まとめ

$j_z = l - \frac{1}{2}$ を持つ状態

$Y_{ll}\chi_\downarrow$ と $Y_{l\,l-1}\chi_\uparrow$

適当な線形結合をとることによって

$$\begin{cases} |l+\frac{1}{2}, l-\frac{1}{2}\rangle = \sqrt{\frac{2l}{2l+1}}\, Y_{l\,l-1}\chi_\uparrow + \sqrt{\frac{1}{2l+1}}\, Y_{ll}\chi_\downarrow \\ |l-\frac{1}{2}, l-\frac{1}{2}\rangle = \sqrt{\frac{1}{2l+1}}\, Y_{l\,l-1}\chi_\uparrow - \sqrt{\frac{2l}{2l+1}}\, Y_{ll}\chi_\downarrow \end{cases}$$

一般に

$$|l+\tfrac{1}{2}, m\rangle = \alpha\, Y_{l\,m-\frac{1}{2}}\chi_\uparrow + \beta\, Y_{l\,m+\frac{1}{2}}\chi_\downarrow$$
$$|l-\tfrac{1}{2}, m\rangle = \beta \quad \because \quad -\alpha \quad \because$$

或いは

$$|jm\rangle = \sum_{m_l, m_s} \underbrace{\langle l\, m_l\, \tfrac{1}{2}\, m_s | jm\rangle}_{\text{クレブシュ・ゴルダン係数}} Y_{lm_l}\chi_{m_s}$$

$$\vec{j} = \vec{l}^{\,2} + \vec{s}^{\,2} + 2\vec{l}\cdot\vec{s}$$

$$\rightsquigarrow \quad \vec{l}\cdot\vec{s} = \frac{1}{2}\left(\vec{j}^{\,2} - \vec{l}^{\,2} - \vec{s}^{\,2}\right)$$

$$\rightsquigarrow \quad \vec{l}\cdot\vec{s}\,|jm\rangle = \underbrace{\frac{1}{2}\left[j(j+1) - l(l+1) - \frac{3}{4}\right]}\,|jm\rangle$$

$j = l + \frac{1}{2}$:

$$\frac{1}{2}\left[(l+\tfrac{1}{2})(l+\tfrac{3}{2}) - l(l+1) - \tfrac{3}{4}\right]$$

$$= \frac{1}{2}\left(\cancel{l^2} + 2l + \cancel{\tfrac{3}{4}} - \cancel{l^2} - l - \cancel{\tfrac{3}{4}}\right) = \frac{l}{2}$$

$j = l - \frac{1}{2}$:

$$\frac{1}{2}\left[(l-\tfrac{1}{2})(l+\tfrac{1}{2}) - l(l+1) - \tfrac{3}{4}\right]$$

$$= \frac{1}{2}\left(l^2 - \tfrac{1}{4} - l^2 - l - \tfrac{3}{4}\right) = -\frac{1}{2}(l+1)$$