# ▨ 角運動量の合成（スピン-軌道力）

## ◦ 複習

$[l_i, l_j] = i\varepsilon_{ijk} l_k$

$[l_x, l_y] = i l_z$
$[l_y, l_z] = i l_x$
$[l_z, l_x] = i l_y$

$l_\pm \equiv l_x \pm i l_y$

$l_\pm |lm\rangle = \sqrt{(l \mp m)(l \pm m + 1)}\ |l\, m \pm 1\rangle$

## ◦ l-s 力

$\vec{j} = \vec{l} + \vec{s}$

(note) $\vec{j}^2 = \vec{l}^2 + \vec{s}^2 + 2\vec{l}\cdot\vec{s}$
$= \vec{l}^2 + \vec{s}^2 + 2 l_z s_z + l_+ s_- + l_- s_+$

角度成分とスピン部分の積 ： $Y_{lm}\chi_{m_s}$

| $m_j = m_l + m_s$ | | |
|---|---|---|
| $l + \frac{1}{2}$ | $Y_{ll}\chi_\uparrow$ | |
| $l - \frac{1}{2}$ | $Y_{l\,l-1}\chi_\uparrow$ | $Y_{ll}\chi_\downarrow$ |
| $l - \frac{3}{2}$ | $Y_{l\,l-2}\chi_\uparrow$ | $Y_{l\,l-1}\chi_\downarrow$ |
| ⋮ | | |
| $-l - \frac{1}{2}$ | $Y_{l\,-l}\chi_\downarrow$ | |

$$\vec{j}^{\,2}\, Y_{\ell\ell}\,\chi_\uparrow = \left(\vec{l}^{\,2} + \vec{s}^{\,2} + 2l_z s_z + \cancel{l_+ s_-} + \cancel{l_- s_+}\right) Y_{\ell\ell}\,\chi_\uparrow$$

$$= \Big[\underbrace{\ell(\ell+1) + \tfrac{3}{4} + \ell}_{\ell^2 + 2\ell + \frac{3}{4} \,=\, (\ell+\frac{1}{2})(\ell+\frac{3}{2})}\Big]\, Y_{\ell\ell}\,\chi_\uparrow$$

$$\therefore \quad Y_{\ell\ell}\,\chi_\uparrow = |\ell+\tfrac{1}{2}, \ell+\tfrac{1}{2}\rangle$$

$$(l_- + s_-)\, Y_{\ell\ell}\,\chi_\uparrow = j_-\, |\ell+\tfrac{1}{2}, \ell+\tfrac{1}{2}\rangle$$

$$= \sqrt{(\ell+\tfrac{1}{2} + \ell+\tfrac{1}{2})(\cancel{\ell+\tfrac{1}{2} - \ell - \tfrac{1}{2}} + 1)} \times |\ell+\tfrac{1}{2}, \ell-\tfrac{1}{2}\rangle$$

$$= \sqrt{2\ell+1}\, |\ell+\tfrac{1}{2}, \ell-\tfrac{1}{2}\rangle$$

$$= \sqrt{2\ell}\, \underline{Y_{\ell\ell-1}\,\chi_\uparrow} + \underline{Y_{\ell\ell}\,\chi_\downarrow}$$

$$\therefore \quad \boxed{|\ell+\tfrac{1}{2}, \ell-\tfrac{1}{2}\rangle = \sqrt{\frac{2\ell}{2\ell+1}}\, Y_{\ell\ell-1}\,\chi_\uparrow + \frac{1}{\sqrt{2\ell+1}}\, Y_{\ell\ell}\,\chi_\downarrow}$$

これと直交する状態を作ってみると：

$$\frac{1}{\sqrt{2\ell+1}}\, Y_{\ell\ell-1}\,\chi_\uparrow - \sqrt{\frac{2\ell}{2\ell+1}}\, Y_{\ell\ell}\,\chi_\downarrow$$

(note) $\vec{j}^{\,2}\left(Y_{\ell\ell-1}\,\chi_\uparrow - \sqrt{2\ell}\, Y_{\ell\ell}\,\chi_\downarrow\right)$

$$= \left(\vec{l}^{\,2} + \vec{s}^{\,2} + 2l_z s_z + l_+ s_- + l_- s_+\right)\left(Y_{\ell\ell-1}\,\chi_\uparrow - \sqrt{2\ell}\, Y_{\ell\ell}\,\chi_\downarrow\right)$$

$$= \left( l(l+1) + \frac{3}{4} + 2(l-1)\cdot\frac{1}{2} \right) Y_{l\,l-1}\chi_\uparrow + \sqrt{(l-l+1)(l+l-1+1)}\,\sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)} \times Y_{ll}\chi_\downarrow$$

$$- \sqrt{2l}\left( l(l+1) + \frac{3}{4} + 2l\cdot\left(-\frac{1}{2}\right) \right) Y_{ll}\chi_\downarrow - \sqrt{2l}\,\sqrt{2l}\, Y_{l\,l-1}\chi_\uparrow$$

$$= \left( l^2 + l + \frac{3}{4} + l - 1 - 2l \right) Y_{l\,l-1}\chi_\uparrow$$

$$- \sqrt{2l}\left( l^2 + l - l + \frac{3}{4} - 1 \right) Y_{ll}\chi_\downarrow$$

$$= \left( l^2 - \frac{1}{4} \right)\left( Y_{l\,l-1}\chi_\uparrow - \sqrt{2l}\, Y_{ll}\chi_\downarrow \right)$$

$$\Downarrow$$

$$\left| l-\frac{1}{2}, l-\frac{1}{2} \right\rangle = \frac{1}{\sqrt{2l+1}} Y_{l\,l-1}\chi_\uparrow - \sqrt{\frac{2l}{2l+1}}\, Y_{ll}\chi_\downarrow$$

まとめ

$j_z = l - \frac{1}{2}$ を持つ状態

$Y_{ll}\chi_\downarrow$ と $Y_{l\,l-1}\chi_\uparrow$

適当な線形結合をとることによって

$$\begin{cases} |l+\frac{1}{2}, l-\frac{1}{2}\rangle = \sqrt{\frac{2l}{2l+1}}\, Y_{l\,l-1}\chi_\uparrow + \sqrt{\frac{1}{2l+1}}\, Y_{ll}\chi_\downarrow \\ |l-\frac{1}{2}, l-\frac{1}{2}\rangle = \sqrt{\frac{1}{2l+1}}\, Y_{l\,l-1}\chi_\uparrow - \sqrt{\frac{2l}{2l+1}}\, Y_{ll}\chi_\downarrow \end{cases}$$

一般に

$$|l+\tfrac{1}{2}, m\rangle = \alpha\, Y_{l\,m-\frac{1}{2}}\chi_\uparrow + \beta\, Y_{l\,m+\frac{1}{2}}\chi_\downarrow$$

$$|l-\tfrac{1}{2}, m\rangle = \beta \quad 〃 \quad -\alpha \quad 〃$$

或いは

$$|j\,m\rangle = \sum_{m_l, m_s} \underbrace{\langle l\, m_l\, \tfrac{1}{2}\, m_s | j\, m\rangle}_{\text{クレブシュ・ゴルダン係数}} Y_{l m_l}\chi_{m_s}$$

(note)

$$j^2 \,|Y_{l\,m-\frac{1}{2}}\chi_\uparrow\rangle = \left[l(l+1) + \frac{3}{4} + 2\left(m-\frac{1}{2}\right)\cdot\frac{1}{2}\right] |Y_{l\,m-\frac{1}{2}}\chi_\uparrow\rangle$$

$$+ \sqrt{\left(l-m+\frac{1}{2}\right)\left(l+m-\frac{1}{2}+1\right)}\; |Y_{l\,m+\frac{1}{2}}\chi_\downarrow\rangle$$

$$= \left[l^2 + l + m + \frac{1}{4}\right] |Y_{l\,m-\frac{1}{2}}\chi_\uparrow\rangle$$

$$+ \sqrt{\left(l+\frac{1}{2}\right)^2 - m^2}\; |Y_{l\,m+\frac{1}{2}}\chi_\downarrow\rangle$$

$$j^2 \,|Y_{l\,m+\frac{1}{2}}\chi_\downarrow\rangle = \left[l(l+1) + \frac{3}{4} + 2\left(m+\frac{1}{2}\right)\left(-\frac{1}{2}\right)\right] |Y_{l\,m+\frac{1}{2}}\chi_\downarrow\rangle$$

$$+ \sqrt{\left(l+\frac{1}{2}\right)^2 - m^2}\; |Y_{l\,m-\frac{1}{2}}\chi_\uparrow\rangle$$

$$= \left(l^2 + l - m + \frac{1}{4}\right) |Y_{l\,m+\frac{1}{2}}\chi_\downarrow\rangle$$

$$+ \sqrt{\left(l+\frac{1}{2}\right)^2 - m^2}\; |Y_{l\,m-\frac{1}{2}}\chi_\uparrow\rangle$$

$\therefore$

$$\begin{array}{cc}
j^2 = \begin{pmatrix} l^2 + l + m + \frac{1}{4} & \sqrt{\left(l+\frac{1}{2}\right)^2 - m^2} \\ \sqrt{\left(l+\frac{1}{2}\right)^2 - m^2} & l^2 + l - m + \frac{1}{4} \end{pmatrix} & \begin{array}{c} |Y_{l\,m-\frac{1}{2}}\chi_\uparrow\rangle \\ |Y_{l\,m+\frac{1}{2}}\chi_\downarrow\rangle \end{array} \\
\quad\quad |Y_{l\,m-\frac{1}{2}}\chi_\uparrow\rangle \quad\quad |Y_{l\,m+\frac{1}{2}}\chi_\downarrow\rangle &
\end{array}$$

$\longrightarrow$ 対角化

$$\begin{pmatrix} l^2+l+m+\frac{1}{4} & \sqrt{(l+\frac{1}{2})^2-m^2} \\ \sqrt{(l+\frac{1}{2})^2-m^2} & l^2+l-m+\frac{1}{4} \end{pmatrix} \begin{pmatrix} \sqrt{l+m+\frac{1}{2}} \\ \sqrt{l-m+\frac{1}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} (l^2+l+m+\frac{1}{4})\sqrt{l+m+\frac{1}{2}} + (l-m+\frac{1}{2})\sqrt{l+m+\frac{1}{2}} \\ (l+m+\frac{1}{2})\sqrt{l-m+\frac{1}{2}} + (l^2+l-m+\frac{1}{4})\sqrt{l-m+\frac{1}{2}} \end{pmatrix}$$

$$= \underbrace{\left(l^2+2l+\frac{3}{4}\right)}_{\substack{\parallel \\ (l+\frac{1}{2})(l+\frac{3}{2})}} \begin{pmatrix} \sqrt{l+m+\frac{1}{2}} \\ \sqrt{l-m+\frac{1}{2}} \end{pmatrix}$$

$\downarrow$

$$\langle l \; m-\tfrac{1}{2} \; \tfrac{1}{2} \; \tfrac{1}{2} \mid l+\tfrac{1}{2} \; m \rangle = \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}$$

$$\langle l \; m+\tfrac{1}{2} \; \tfrac{1}{2} \; -\tfrac{1}{2} \mid l+\tfrac{1}{2} \; m \rangle = \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}$$

$$\begin{pmatrix} l^2+l+m+\frac{1}{4} & \sqrt{(l+\frac{1}{2})^2-m^2} \\ \sqrt{(l+\frac{1}{2})^2-m^2} & l^2+l-m+\frac{1}{4} \end{pmatrix} \begin{pmatrix} -\sqrt{l-m+\frac{1}{2}} \\ \sqrt{l+m+\frac{1}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -(l^2+l+m+\frac{1}{4})\sqrt{l-m+\frac{1}{2}} + (l+m+\frac{1}{2})\sqrt{l-m+\frac{1}{2}} \\ -(l-m+\frac{1}{2})\sqrt{l+m+\frac{1}{2}} + (l^2+l-m+\frac{1}{4})\sqrt{l+m+\frac{1}{2}} \end{pmatrix}$$

$$= \underbrace{\left(l^2-\frac{1}{4}\right)}_{=(l-\frac{1}{2})(l+\frac{1}{2})} \begin{pmatrix} -\sqrt{l-m+\frac{1}{2}} \\ \sqrt{l+m+\frac{1}{2}} \end{pmatrix}$$

$$\langle l\ \ m-\tfrac{1}{2}\ \ \tfrac{1}{2}\ \ \tfrac{1}{2} \mid l-\tfrac{1}{2}\ \ m \rangle = -\sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}$$

$$\langle l\ \ m+\tfrac{1}{2}\ \ \tfrac{1}{2}\ \ -\tfrac{1}{2} \mid l-\tfrac{1}{2}\ \ m \rangle = \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}$$

$$\vec{j}^2 = \vec{l}^2 + \vec{s}^2 + 2\vec{l}\cdot\vec{s}$$

$$\Rightarrow \quad \vec{l}\cdot\vec{s} = \frac{1}{2}\left(\vec{j}^2 - \vec{l}^2 - \vec{s}^2\right)$$

$$\Rightarrow \quad \vec{l}\cdot\vec{s}\,|jm\rangle = \underbrace{\frac{1}{2}\left[j(j+1) - l(l+1) - \frac{3}{4}\right]}\,|jm\rangle$$

$j = l + \frac{1}{2}$:

$$\frac{1}{2}\left[(l+\tfrac{1}{2})(l+\tfrac{3}{2}) - l(l+1) - \tfrac{3}{4}\right]$$

$$= \frac{1}{2}\left(\cancel{l^2} + 2l + \cancel{\tfrac{3}{4}} - \cancel{l^2} - l - \cancel{\tfrac{3}{4}}\right) = \frac{l}{2}$$

$j = l - \frac{1}{2}$:

$$\frac{1}{2}\left[(l-\tfrac{1}{2})(l+\tfrac{1}{2}) - l(l+1) - \tfrac{3}{4}\right]$$

$$= \frac{1}{2}\left(l^2 - \tfrac{1}{4} - l^2 - l - \tfrac{3}{4}\right) = -\frac{1}{2}(l+1)$$