**Pairing correlations**

$l \quad \rightarrow \quad 0^+, 2^+, 4^+, 6^+, \ldots$

- Without pairing correlation
- With pairing correlation

Spin and parity of the ground state of nuclei

- even-even nuclei: always $0^+$ (no exception)
Simple interpretation:

The spatial overlap is the largest for the $I=0$ pair.

"Pairing Correlation"
Di-neutron correlation

What is the spatial structure of the two-valence neutrons?

If the two neutrons moved independently, one neutron does not care where the other neutron is.

How does this change due to the pairing correlation?
Three-body model: microscopic understanding of di-neutron correlation

\[ H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V_{nC}(r_1) + V_{nC}(r_2) + V_{nn}(r_1, r_2) + \frac{P_{\text{core}}^2}{2A_cm} \]

the ground state of this three-body Hamiltonian and also the density distribution

(e.g.,) expand the \( \psi \) with the eigen-functions for \( H \) without \( V_{nn} \) and determine the expansion coefficients

\[
\psi_{gs}(r_1, r_2) = A \sum_{nn'l'j} \alpha_{nn'l'j} \psi^{(2)}_{nn'l'j}(r_1, r_2)
\]

\[
\psi^{(2)}_{nn'l'j}(r_1, r_2) = \sum_m \langle jmjj - m | 00 \rangle \psi_{nljm}(r_1) \psi_{n'l'j-m}(r_2)
\]
Comparison between with and without paring correlations

$^{11}\text{Li}$ a distribution of one of the neutrons when the other neutron is at $(z_1, x_1) = (3.4 \text{ fm}, 0)$

- When no pairing, symmetric between $z$ and $-z$. The distribution does not change wherever the 2nd neutron is.
- When with pairing, the nearside density is enhanced. The distribution changes when the 2nd neutron moves.
What is Di-neutron correlation?

Correlation: \( \langle AB \rangle \neq \langle A \rangle \langle B \rangle \)

Example: \( ^{18}\text{O} = ^{16}\text{O} + n + n \)

cf. \( ^{16}\text{O} + n \): 3 bound states (\(1d_{5/2}, 2s_{1/2}, 1d_{3/2}\))

i) Without nn interaction: \( |nn\rangle = |(1d_{5/2})^2\rangle \)

Distribution of the 2\(^{nd}\) neutron when the 1\(^{st}\) neutron is at \(z_1\):

- Two neutrons move independently
- No influence of the 2\(^{nd}\) neutron from the 1\(^{st}\) neutron

\[ \langle AB \rangle = \langle A \rangle \langle B \rangle \]
What is Di-neutron correlation?

Example: $^{18}\text{O} = ^{16}\text{O} + n + n$

cf. $^{16}\text{O} + n$: 3 bound states ($1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$)

ii) nn interaction: works only on the positive parity (bound) states

$$|nn\rangle = \alpha|1d_{5/2}\rangle^2 + \beta|2s_{1/2}\rangle^2 + \gamma|1d_{3/2}\rangle^2$$

✓ distribution changes according to the 1$^{\text{st}}$ neutron (nn correlation)
✓ but, the distribution of the 2$^{\text{nd}}$ neutron has peaks both at $z_1$ and $-z_1$

→ this is NOT called the di-neutron correlation
What is Di-neutron correlation?  

Example: $^{18}\text{O} = ^{16}\text{O} + n + n$

cf. $^{16}\text{O} + n$: 3 bound states ($1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$)

iii) $nn$ interaction: works also on the continuum states

$$|nn\rangle = \sum_{n,n',j,l} C_{nn'jl} |(nn'jl)^2\rangle$$

✓ spatial correlation: the density of the $2^{nd}$ neutron localized close to the $1^{st}$ neutron (dineutron correlation)

✓ parity mixing: essential role

cf. F. Catara et al., PRC29(‘84)1091
dineutron correlation: caused by the admixture of different parity states

\[ \left( 0h_{11/2} \right)^2 + \left( 0i_{13/2} \right)^2 \]

F. Catara, A. Insolia, E. Maglione, and A. Vitturi, PRC29('84)1091

interference of even and odd partial waves

\[
\rho_2(x_1, x_2) = |\Psi_{ee}(x_1, x_2)|^2 + |\Psi_{oo}(x_1, x_2)|^2 \\
+ 2\Psi_{ee}(x_1, x_2)\Psi_{oo}(x_1, x_2)
\]
spatial localization of two neutrons
(dineutron correlation)

cf. Migdal, Soviet J. of Nucl. Phys. 16 (‘73) 238
Bertsch, Broglia, Riedel, NPA91(‘67)123

weakly bound systems

easy to mix different parity states due to the continuum couplings
+ enhancement of pairing on the surface

K.H. and H. Sagawa,
PRC72(‘05)044321
pairing gap in infinite nuclear matter

Matsuo, PRC 73(’06)044309
spatial localization of two neutrons (dineutron correlation)

K.H. and H. Sagawa, PRC72(‘05)044321

easy to mix different parity states due to the continuum couplings

+ enhancement of pairing on the surface

dineutron correlation: enhanced

cf. Migdal, Soviet J. of Nucl. Phys. 16 (‘73) 238
Bertsch, Broglia, Riedel, NPA91(‘67)123

weakly bound systems

parity mixing

cf. - Bertsch, Esbensen, Ann. of Phys. 209(‘91)327
- M. Matsuo, K. Mizuyama, Y. Serizawa,
PRC71(‘05)064326

K.H. and H. Sagawa, PRC72(‘05)044321
The BCS theory

Many-particles in non-degenerate levels
~ mean-field approx. for the pairing channel ~

Simplified pairing interaction

\[ V = -G \, P^\dagger \, P; \quad P^\dagger = \sum_{\nu > 0} a^\dagger_{\nu} a^\dagger_{\nu} \]

\( \bar{\nu} \): the time reversed state of \( \nu \)
e.g.,
\[ |\nu\rangle = |njl m\rangle, \quad |\bar{\nu}\rangle = |njl - m\rangle \]

\( 0^+, 2^+, 4^+, 6^+, \ldots \)

\( P^\dagger = \sum_j \sum_{m > 0} a^\dagger_{jm} a^\dagger_{j-m} \)

: an operator to create an \( I = 0 \) pair

delta force

Cf. Metallic superconductivity

\( 2^+, 4^+, 6^+ \)

monopole pairing force
Solve the pairing Hamiltonian

\[
H = \sum_{\nu} \epsilon_{\nu} (a_{\nu}^{\dagger}a_{\nu} + a_{\nu}^{\dagger}a_{\nu}^{\dagger}) - G \left( \sum_{\nu > 0} a_{\nu}^{\dagger}a_{\nu}^{\dagger} \right) \left( \sum_{\nu > 0} a_{\nu}^{\dagger}a_{\nu} \right)
\]

in the mean-field approximation

- **Mean-field approximation:**

\[
V = -GP^{\dagger}P \rightarrow -G \left( \langle P^{\dagger} \rangle P + P^{\dagger}\langle P \rangle \right) = -\Delta (P^{\dagger} + P)
\]

\[
\Delta \equiv G\langle P^{\dagger} \rangle = G\langle P \rangle
\]

particle number violation

we consider \(H' = H - \lambda \hat{N}\) instead of \(H\):

\[
H' \rightarrow \sum_{k>0} (\epsilon_{k} - \lambda)(a_{k}^{\dagger}a_{k} + a_{k}^{\dagger}a_{k}^{\dagger}) - \Delta \sum_{k>0} (a_{k}^{\dagger}a_{k}^{\dagger} + a_{k}^{\dagger}a_{k})
\]
$H' \rightarrow \sum_{k>0} (\epsilon_k - \lambda)(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) - \Delta \sum_{k>0} (a_k^\dagger a_{-k}^\dagger + a_{-k} a_k)$

Bogoliubov transformation

$\alpha_\nu^\dagger = u_\nu a_\nu^\dagger - v_\nu a_\nu$, $\alpha_{\nu}^\dagger = u_\nu a_\nu^\dagger + v_\nu a_\nu$

(Quasi-particle operator)

Transform $H'$ in a form of

$H' = \text{const.} + \sum_{k>0} E_k(\alpha_k^\dagger \alpha_k + \alpha_{-k}^\dagger \alpha_{-k})$

$\text{g.s.}: \alpha_k |BCS\rangle = 0$

$1^{st}$ excited state: $|1_k\rangle = \alpha_k^\dagger |BCS\rangle$ at $E_k$

…. and so on.
Ground state wave function: \[ \alpha_k |\text{BCS}\rangle = 0 \]

\[ |\text{BCS}\rangle = \prod_{\nu > 0} \left( u_\nu + v_\nu a_\nu^\dagger a_{\nu}^\dagger \right) |0\rangle \]

(note) \[ \langle \text{BCS}|a_\nu^\dagger a_\nu|\text{BCS}\rangle = |v_\nu|^2 \quad : \text{occupation probability} \]

(note) \[ E'_{\text{BCS}} = \langle \text{BCS}|H'|\text{BCS}\rangle \sim 2 \sum_{\nu > 0} (\epsilon_\nu - \lambda) v_\nu^2 - \frac{\Delta^2}{G} \]

\[ H' = \text{const.} + \sum_{k > 0} E_k (\alpha_k^\dagger \alpha_k + \alpha_k^\dagger \alpha_k^\dagger) \]

\[ u_\nu^2 = \frac{1}{2} \left( 1 + \frac{\epsilon_\nu - \lambda}{\sqrt{(\epsilon_\nu - \lambda)^2 + \Delta^2}} \right) \]

\[ v_\nu^2 = \frac{1}{2} \left( 1 - \frac{\epsilon_\nu - \lambda}{\sqrt{(\epsilon_\nu - \lambda)^2 + \Delta^2}} \right) \]

\[ E_k = \sqrt{(\epsilon_k - \lambda)^2 + \Delta^2} \]

Self-consistency condition:

\[ \Delta = G \langle \text{BCS}|\hat{P}|\text{BCS}\rangle = G \sum_{\nu > 0} u_\nu v_\nu \]

\[ = \frac{G}{2} \sum_{\nu > 0} \frac{\Delta}{E_\nu} \]

Gap equation
Wave function:

\[ |\psi_{0+}\rangle = |(ll)L = 0\rangle + \sum_{l'} \frac{\langle (l'l')L = 0 | v_{\text{res}} | (ll)L = 0 \rangle}{2\epsilon_l - 2\epsilon_{l'}} |(l'l')L = 0\rangle + \cdots \]

Each orbit is occupied only partially.

cf. BCS theory
i) Trivial solution: always exists

\[ \Delta = 0 \]

\[ v_{\nu}^2 = 1 \quad (\epsilon_{\nu} \leq \lambda) \]

\[ = 0 \quad (\epsilon_{\nu} > \lambda) \]

\[ |\psi\rangle = \prod_{\nu > 0} a_{\nu}^{\dagger} a_{\nu}^{\dagger} |0\rangle \]

\[ G \text{ a/o } N \rightarrow \text{ large} \]

ii) Superfluid solution

\[ \Delta \neq 0 \]

\[ v_{\nu}^2 < 1 \]

\[ |BCS\rangle = \prod_{\nu > 0} \left( u_{\nu} + v_{\nu} a_{\nu}^{\dagger} a_{\nu}^{\dagger} \right) |0\rangle \]

Number fluctuation

Normal-Superfulid phase transition
Quasi-particle excitations

\[ H \sim E_{BCS} + \sum_{\nu} E_{\nu} \alpha_{\nu}^{\dagger} \alpha_{\nu} \]

- g.s. of even-even nuclei: \(|BCS\rangle\)
- One quasi-particle states:
  \[ |\nu_1\rangle = \alpha_{\nu_1}^{\dagger} |BCS\rangle = a_{\nu_1}^{\dagger} \prod_{\nu \neq \nu_1} (u_{\nu} + v_{\nu} a_{\nu}^{\dagger} a_{\nu}^{\dagger}) |0\rangle \]
  Wave function for odd-mass nuclei
  \[ \langle \nu_1 | H | \nu_1 \rangle = \langle H \rangle + E_{\nu_1} \]
- Two quasi-particle states:
  \[ |\nu_1 \nu_2\rangle = \alpha_{\nu_1}^{\dagger} \alpha_{\nu_2}^{\dagger} |BCS\rangle \]
  Excited state of the even-even nuclei
  \[ \langle \nu_1 \nu_2 | H | \nu_1 \nu_2 \rangle - \langle H \rangle = E_{\nu_1} + E_{\nu_2} \geq 2\Delta \quad \text{(energy gap)} \]

(note) no pairing limit:
\[ \alpha_p^{\dagger} \alpha_h^{\dagger} \rightarrow a_p^{\dagger} a_h, \quad E_p + E_h \rightarrow (\epsilon_p - \lambda) + (\lambda - \epsilon_h) \]

(particle-hole excitation)
Figure 6.1. Excitation spectra of the $^{50}$Sn isotopes.
Effects of pairing on moment of inertia

\[ E_I = \frac{I(I + 1)\hbar^2}{2J} \]

G.F. Bertsch, in “Fifty years of nuclear BCS”

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Fig. 9. Excitation energy of the first $2^+$ state in deformed nuclei. The line shows the prediction assuming a rigid rotor.
Even-odd mass difference and pairing gap

\[ B_{\text{pair}} = \Delta \quad (\text{for even} - \text{even}) \]
\[ = 0 \quad (\text{for even} - \text{odd}) \]
\[ = -\Delta \quad (\text{for odd} - \text{odd}) \]

\[ E(N + 2, Z) = E(N, Z) + 2\lambda \]
\[ E(N + 1, Z) = E(N, Z) + \lambda + \Delta \]

\[ -\Delta_n \sim [E(N + 2, Z) - 2E(N + 1, Z) + E(N, Z)]/2 \]

\[ \Delta \sim 12/\sqrt{A} \quad (\text{MeV}) \]

Bohr-Mottelson ('69)
Hartree-Fock-Bogoliubov (HFB) Theory

HF+BCS method: first solve HF, and then solve the gap equation

s.p. wave functions, occupation probabilities, chemical potential, pairing gaps

\[ \psi_k(r), u_k, v_k \]

Hartree-Fock-Bogoliubov (HFB) theory:

both wave functions and occupation probabilities at the same time

\[ U_k(r), V_k(r) \]

cf. weakly bound systems
\[
\begin{pmatrix}
\tilde{h}(r) - \lambda & \tilde{\Delta}(r) \\
\tilde{\Delta}(r)^* & -\tilde{h}(r) + \lambda
\end{pmatrix}
\begin{pmatrix}
U_k(r) \\
V_k(r)
\end{pmatrix} = E_k
\begin{pmatrix}
U_k(r) \\
V_k(r)
\end{pmatrix}
\]

\[
\tilde{h}(r) = -\frac{\hbar^2}{2m} \nabla^2 + V_{HF}(r)
\]

\[
\rho(r) = \sum_k |V_k(r)|^2
\]

\text{\textit{u,v factors }\rightarrow \text{ u, v functions}}
Application of the HFB method

Density of $^{110}\text{Zr}$ (SHFB-SLy4)

A. Blazkiewicz et al., PRC71(’05)054231

Systematics of $\beta_2$ and $S_{2n}$

M.V. Stoitsov et al., PRC68(’03)054312
Deformed drip-line nuclei

$^{34}$Ne, $^{42}$Mg

$^{100}$Zn

M.V. Stoitsov et al., PRC68(’03)054312
potential energy surface for fission process

A. Staszczak, A. Baran, J. Dobaczewski, and W. Nazarewicz, PRC80 (‘09) 014309