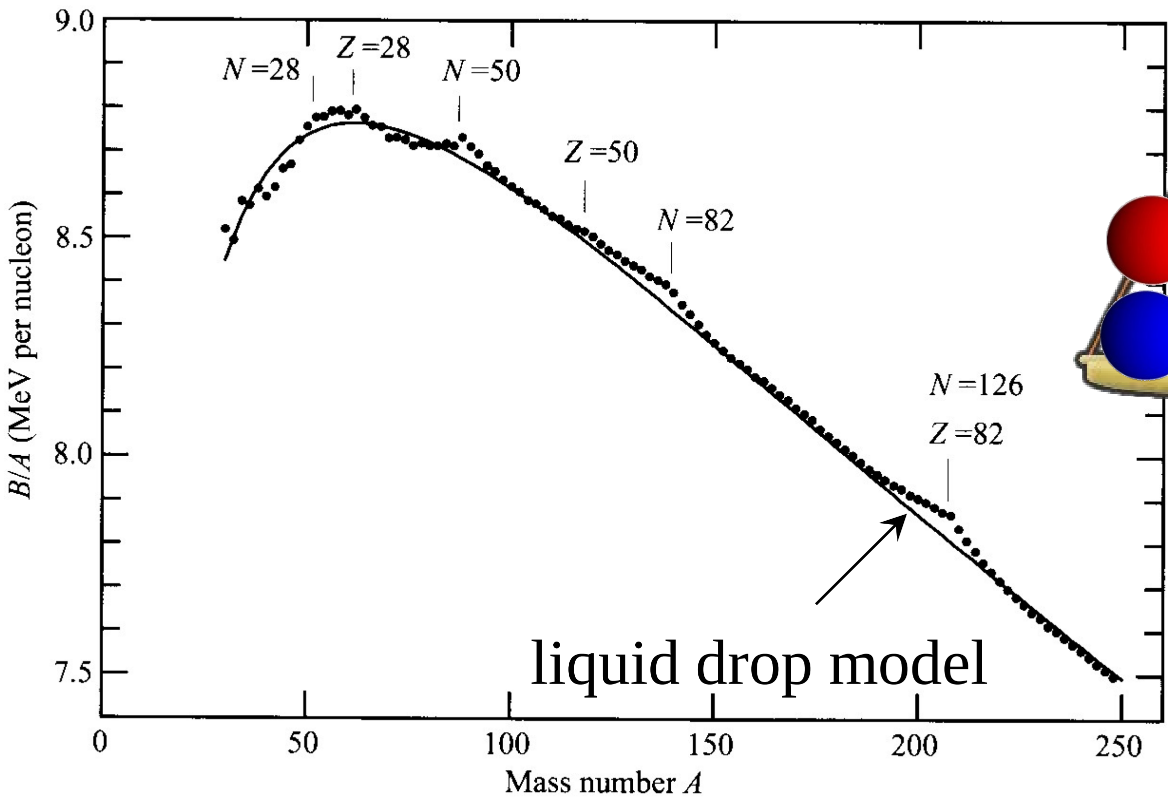
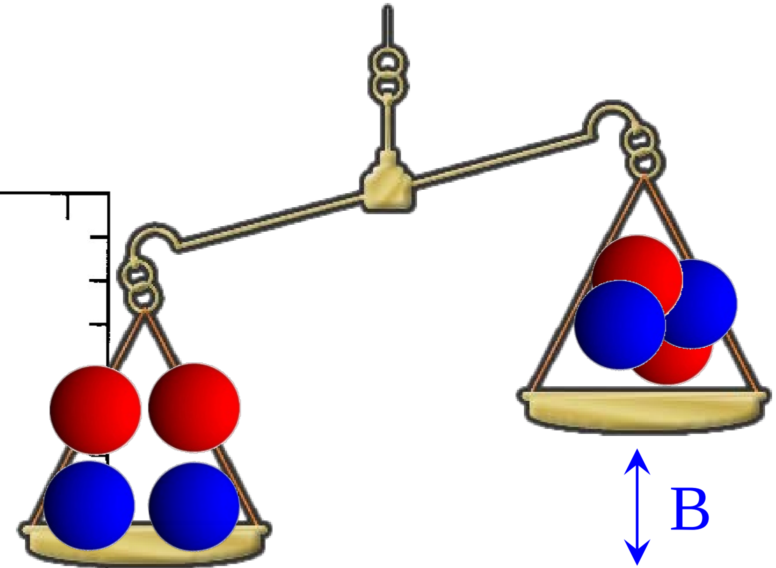
Shell Energy



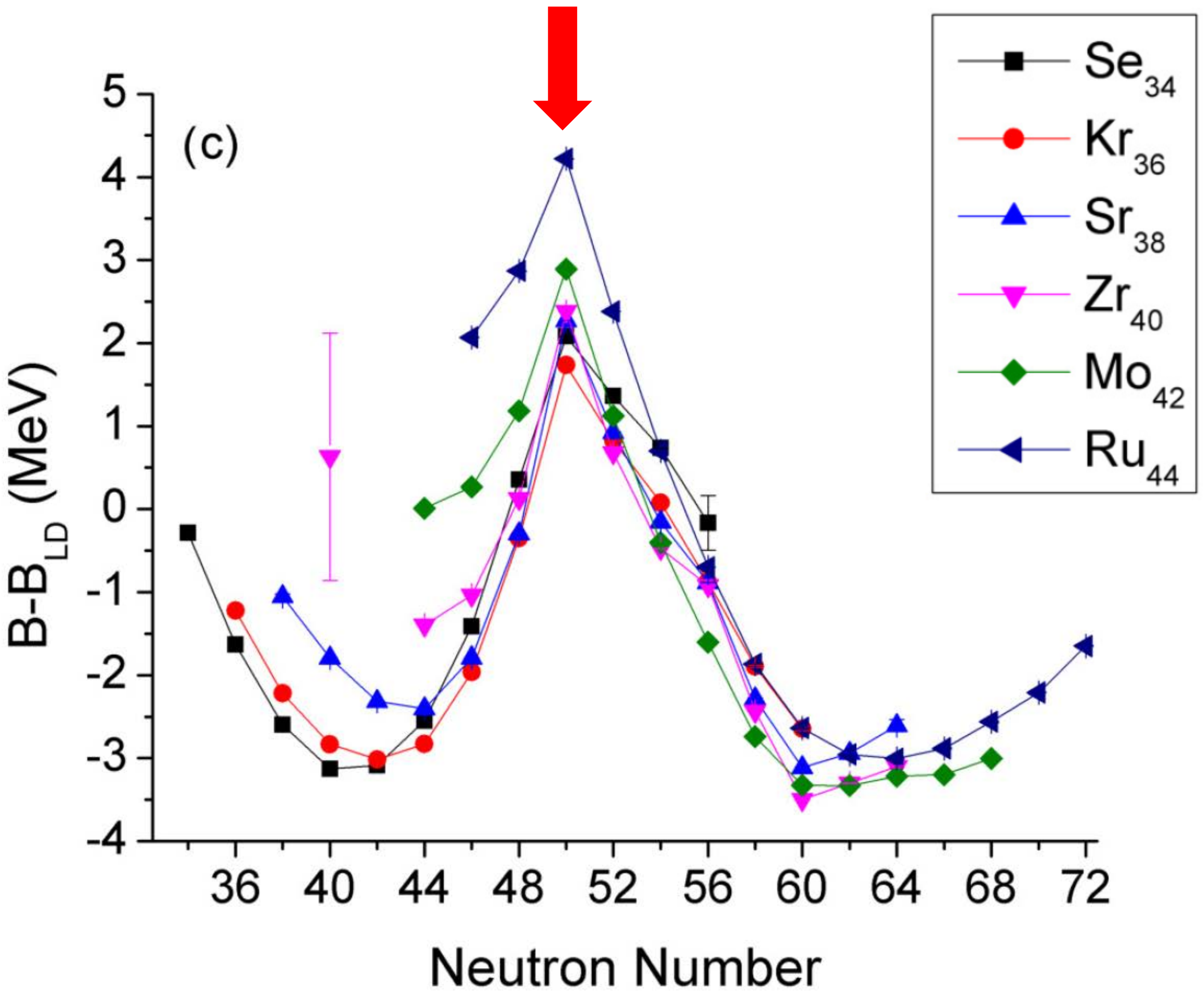


Extra binding for N or Z = 2, 8, 20, 28, 50, 82, 126 (magic numbers)

⇒ Very stable

$^{4}_{2}\mathrm{He}_{2}$, $^{16}_{8}\mathrm{O}_{8}$, $^{40}_{20}\mathrm{Ca}_{20}$, $^{48}_{20}\mathrm{Ca}_{28}$, $^{208}_{82}\mathrm{Pb}_{126}$

N = 50

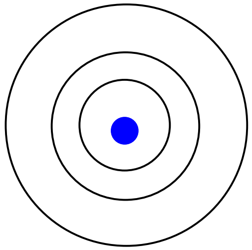


I. Bentley et al., PRC93 ('16) 044337

(note) Atomic magic numbers (Noble gas)

He (Z=2), Ne (Z=10), Ar (Z=18), Kr (Z=36), Xe (Z=54), Rn (Z=86)

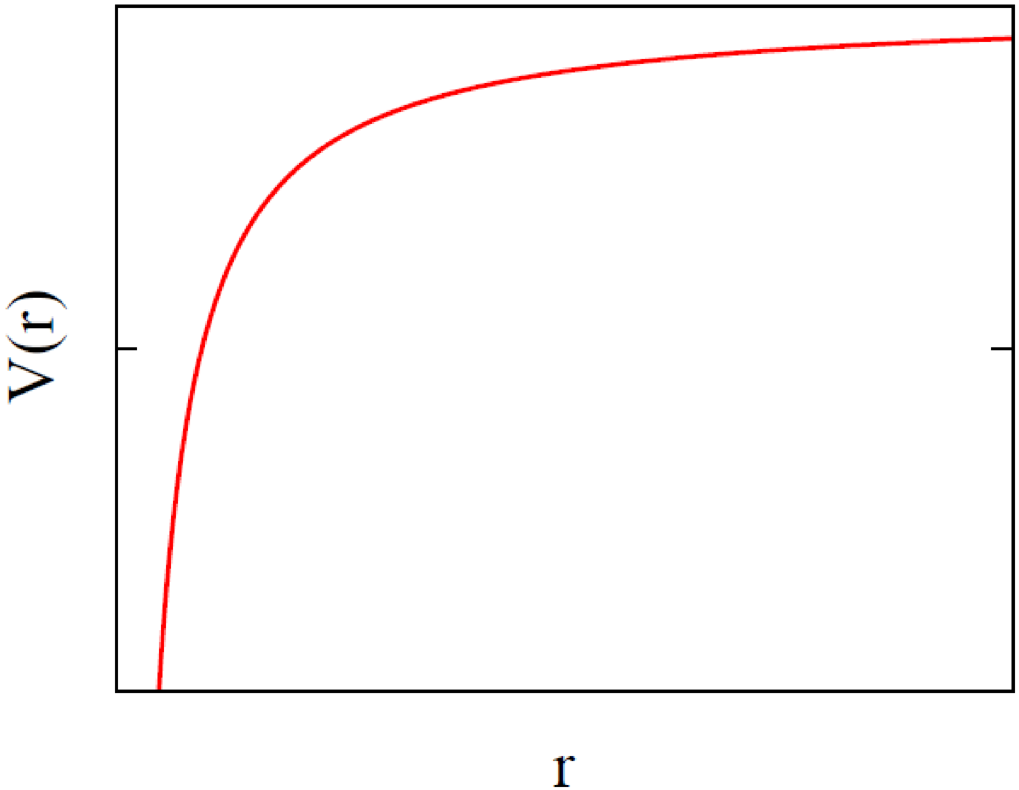
interpretation:



shell structure

magic numbers for electrons

Hydrogen-like potential: $$V(r) = -\frac{Ze^2}{r}$$



$$E_n = -\frac{(Z\alpha)^2}{2n^2} mc^2$$

$$\alpha = \frac{e^2}{\hbar c} \sim \frac{1}{137}$$

$$n = n_r + l + 1$$

[magic numbers for electrons]

Hydrogen-like potential: $$V(r) = -\frac{Ze^2}{r}$$

$$E_n = -\frac{(Z\alpha)^2}{2n^2} mc^2$$

3S 3P 3D

$$\alpha = \frac{e^2}{\hbar c} \sim \frac{1}{137}$$

2S 2P

$$n = n_r + l + 1$$

1S

magic numbers for electrons

Hydrogen-like potential: $$V(r) = -\frac{Ze^2}{r}$$

degeneracy = 2 * (2 *l* +1)

(spin x l_z)

$$E_n = -\frac{(Z\alpha)^2}{2n^2} mc^2$$

3S [2] 3P [6] 3D [10]

$$\alpha = \frac{e^2}{\hbar c} \sim \frac{1}{137}$$

2S [2] 2P [6]

$$n = n_r + l + 1$$

1S [2]

magic numbers for electrons

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(spin x l_z)

$$E_n = -\frac{(Z\alpha)^2}{2n^2} mc^2$$

3S [2]   3P [6]   3D [10]

$$\alpha = \frac{e^2}{\hbar c} \sim \frac{1}{137}$$

2S [2]   2P [6]

$$n = n_r + l + 1$$

1S [2]   ⇒ He

magic numbers for electrons

Hydrogen-like potential: $$V(r) = -\frac{Ze^2}{r}$$

degeneracy = 2 * (2 *l* +1)

(spin x l_z)

$$E_n = -\frac{(Z\alpha)^2}{2n^2} mc^2$$

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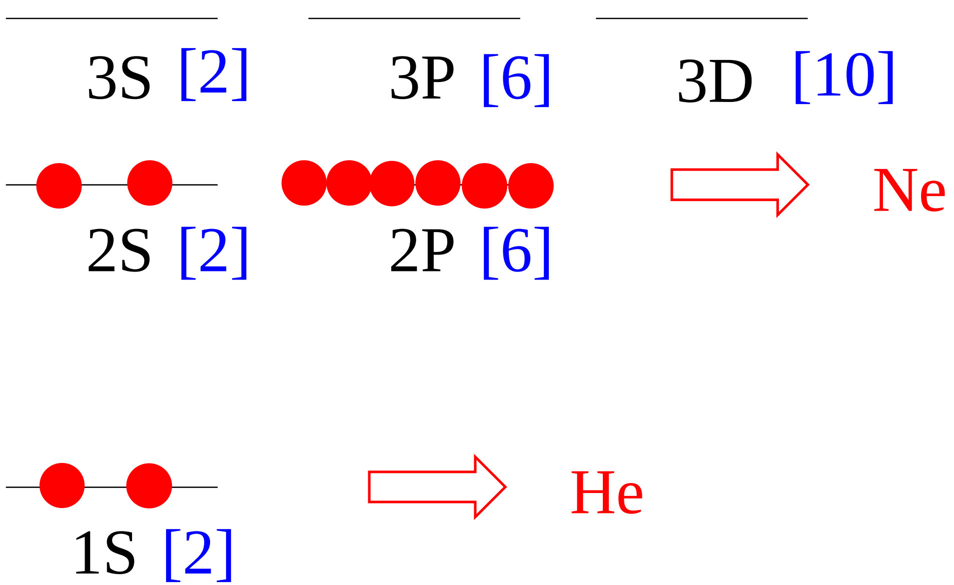
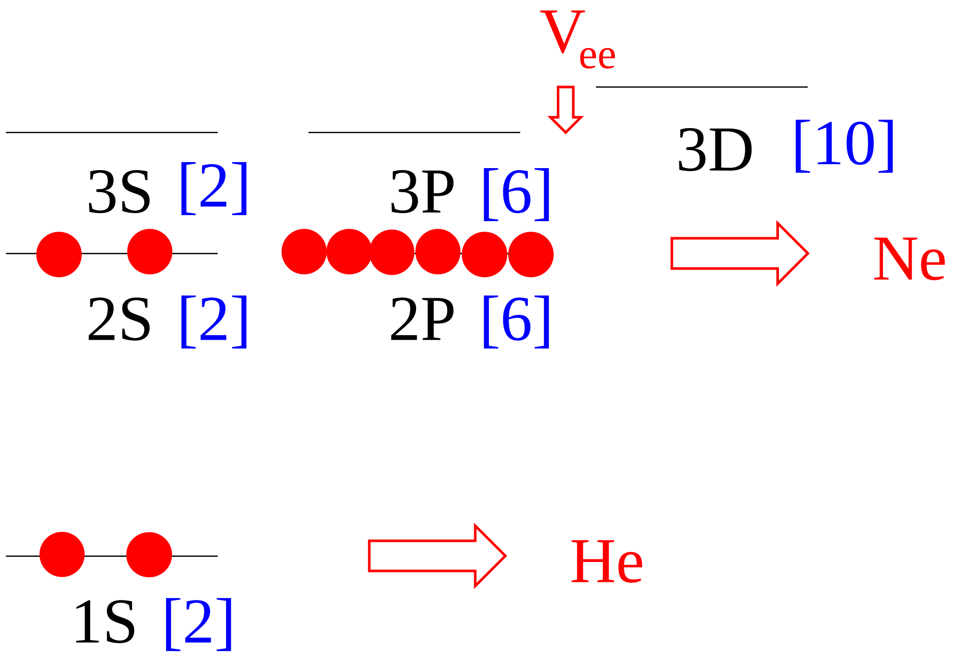


magic numbers for electrons

Hydrogen-like potential: $$V(r) = -\frac{Ze^2}{r}$$

degeneracy = 2 * (2 l +1)



$$E_n = -\frac{(Z\alpha)^2}{2n^2} mc^2$$

$$\alpha = \frac{e^2}{\hbar c} \sim \frac{1}{137}$$

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magic numbers for electrons

Hydrogen-like potential:

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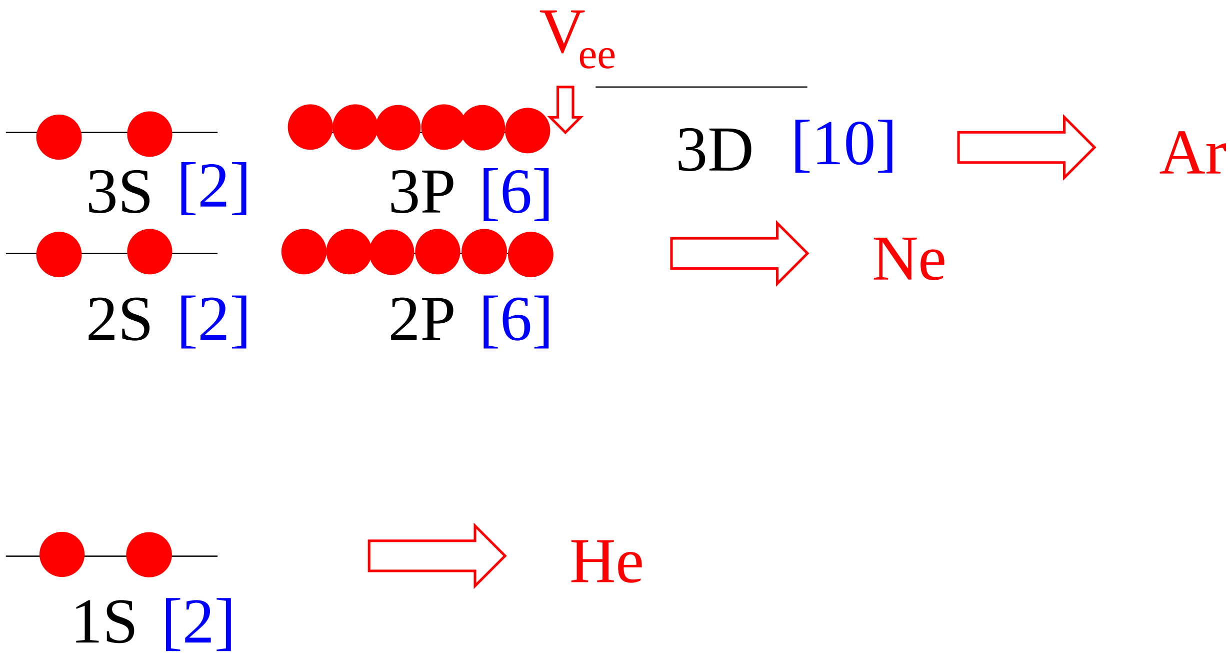


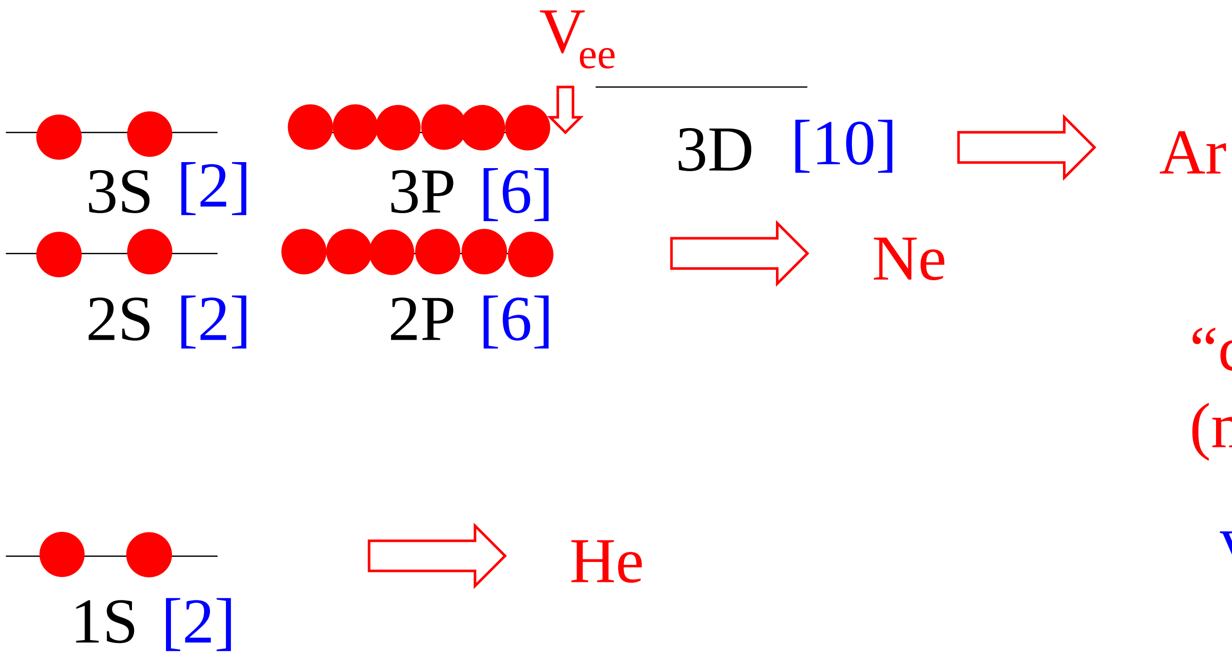
$$\alpha = \frac{e^2}{\hbar c} \sim \frac{1}{137}$$

$$n = n_r + l + 1$$

magic numbers for electrons

Hydrogen-like potential: $$V(r) = -\frac{Ze^2}{r}$$

degeneracy = 2 * (2 *l* +1)

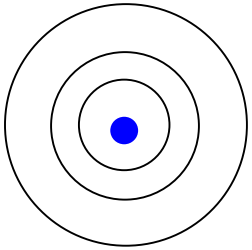


"closed shell"
(magic numbers)

very stable

(note) Atomic magic numbers (Noble gas)

He (Z=2), Ne (Z=10), Ar (Z=18), Kr (Z=36), Xe (Z=54), Rn (Z=86)

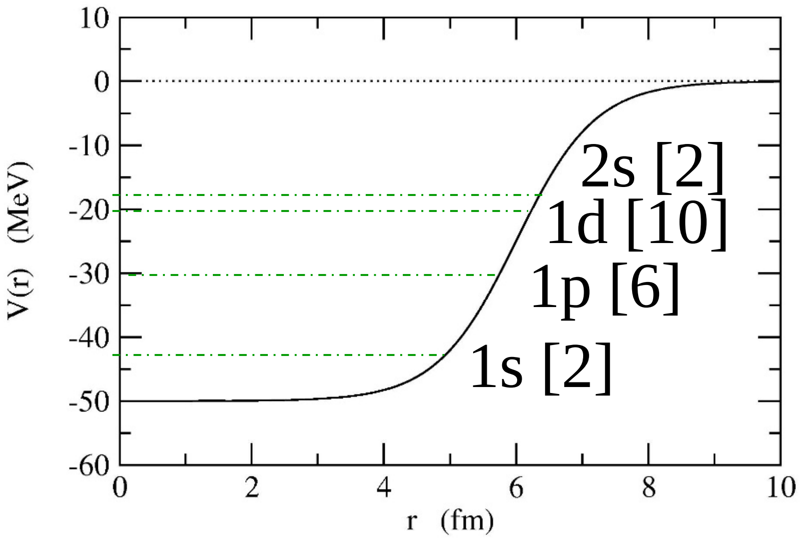


Shell structure

A similar attempt in nuclear physics: independent particle motion in a potential well

Woods-Saxon potential

$$V(r) = \frac{-V_0}{1+\exp[(r-R_0)/a]}$$

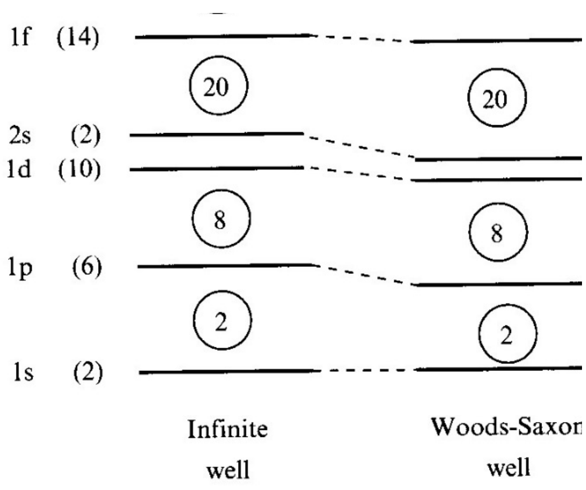


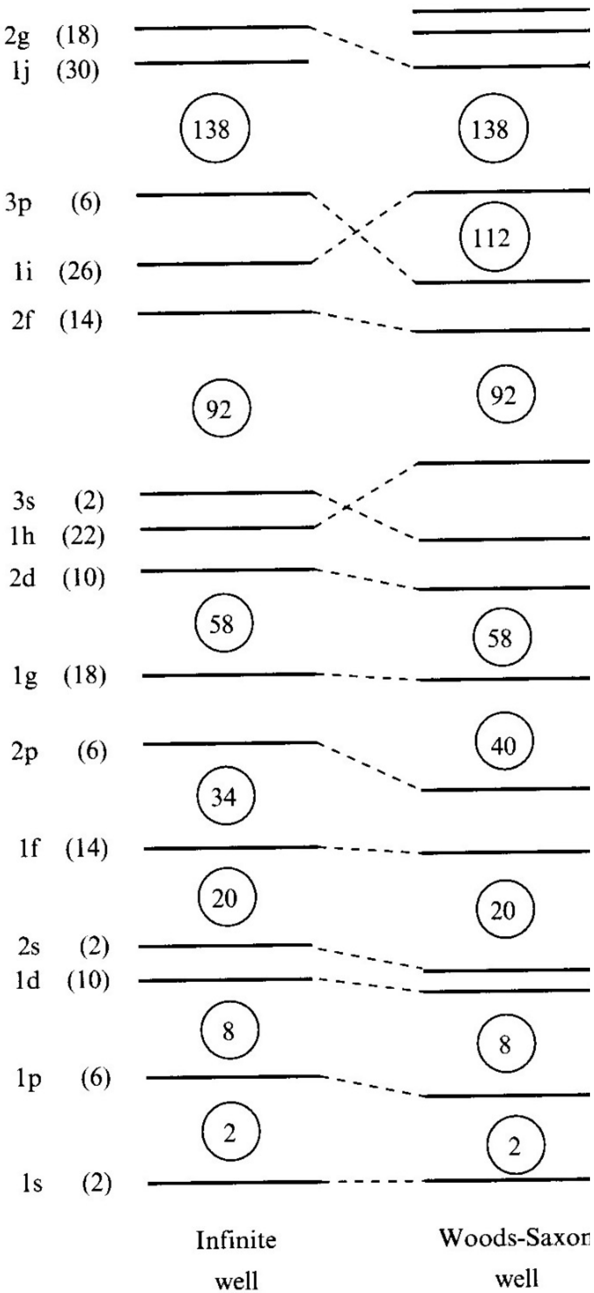
$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) - \epsilon\right]\psi(\boldsymbol{r}) = 0$$

$$\psi(\boldsymbol{r}) = \frac{u_l(r)}{r}Y_{lm}(\hat{\boldsymbol{r}})\cdot\chi_{m_s}$$

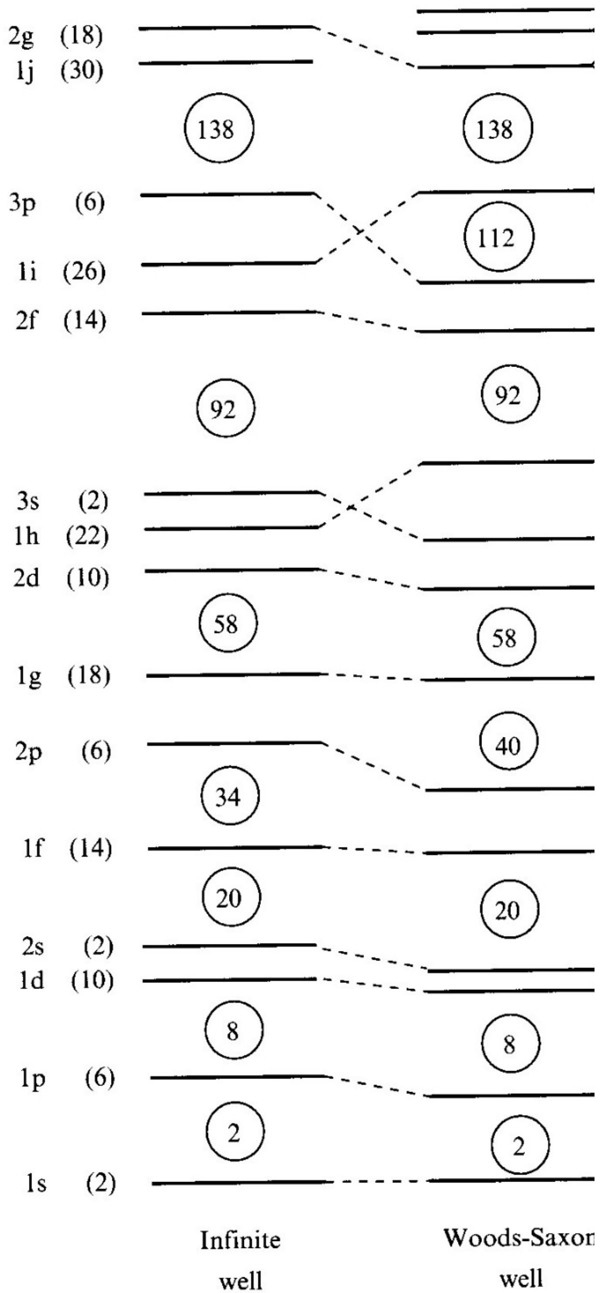
degeneracy: 2*(2l+1)

Nuclear magic numbers:
2, 8, 20, 28, 50, 82, 126





Nuclear magic numbers:
2, 8, 20, 28, 50, 82, 126



Woods-Saxon itself does not provide the correct magic numbers (2,8,20,28, 50,82,126).

Mayer and Jensen (1949):

Strong spin-orbit interaction

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) + V_{ls}(r)\boldsymbol{l}\cdot\boldsymbol{s} - \epsilon\right]\psi(\boldsymbol{r}) = 0$$

$$V_{ls}(r) \sim -\lambda \frac{1}{r}\frac{dV}{dr} \qquad (\lambda > 0)$$

jj coupling shell model

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) - \epsilon\right]\psi(\boldsymbol{r}) = 0 \Longrightarrow \psi_{lmm_s}(\boldsymbol{r}) = \frac{u_l(r)}{r}Y_{lm}(\hat{\boldsymbol{r}})\cdot\chi_{m_s}$$

Spin-orbit interaction

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) + V_{ls}(r)\boldsymbol{l}\cdot\boldsymbol{s} - \epsilon\right]\psi(\boldsymbol{r}) = 0$$

(note) $\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s} \Longrightarrow \boldsymbol{l}\cdot\boldsymbol{s} = \frac{1}{2}(\boldsymbol{j}^2 - \boldsymbol{l}^2 - \boldsymbol{s}^2)$

$$\boxed{\begin{aligned}\psi_{jlm}(\boldsymbol{r}) &= \frac{u_{jl}(r)}{r}\mathcal{Y}_{jlm}(\hat{\boldsymbol{r}}) \\ \mathcal{Y}_{jlm}(\hat{\boldsymbol{r}}) &= \sum_{m_l,m_s}\langle l\ m_l\ 1/2\ m_s|j\ m\rangle Y_{lm_l}(\hat{\boldsymbol{r}})\chi_{m_s}\end{aligned}}$$

$$\begin{aligned}\boldsymbol{j}^2\,\mathcal{Y}_{jlm}(\hat{\boldsymbol{r}}) &= j(j+1)\mathcal{Y}_{jlm}(\hat{\boldsymbol{r}}) \\ j_z\,\mathcal{Y}_{jlm}(\hat{\boldsymbol{r}}) &= m\mathcal{Y}_{jlm}(\hat{\boldsymbol{r}}) \\ \boldsymbol{l}^2\,\mathcal{Y}_{jlm}(\hat{\boldsymbol{r}}) &= l(l+1)\mathcal{Y}_{jlm}(\hat{\boldsymbol{r}}) \\ \boldsymbol{s}^2\,\mathcal{Y}_{jlm}(\hat{\boldsymbol{r}}) &= \frac{1}{2}\left(\frac{1}{2}+1\right)\mathcal{Y}_{jlm}(\hat{\boldsymbol{r}})\end{aligned}$$

jj coupling shell model

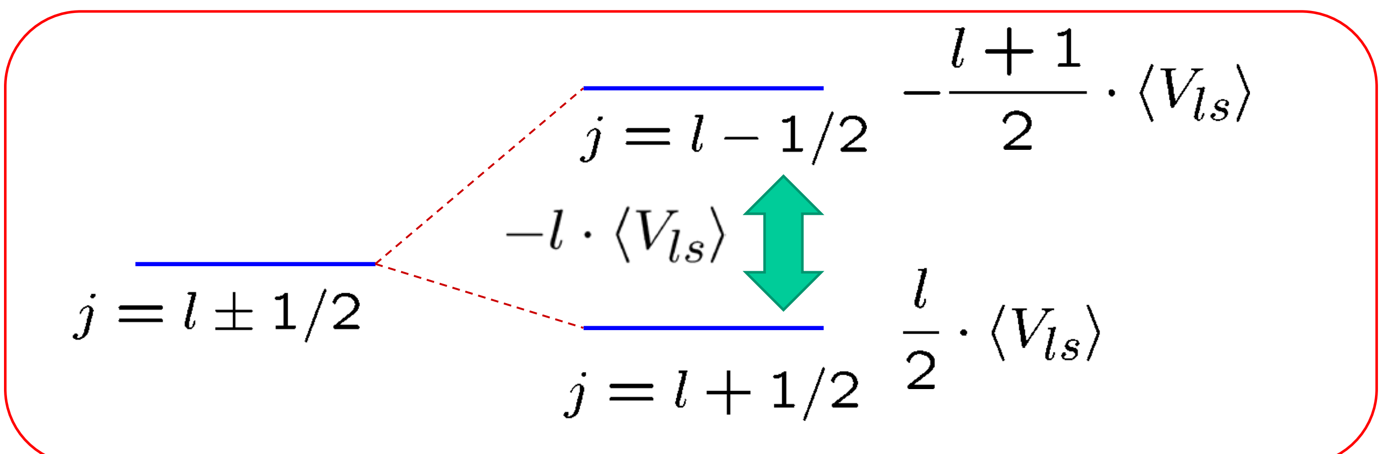
$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) - \epsilon\right]\psi(\boldsymbol{r}) = 0 \Longrightarrow \psi_{lmm_s}(\boldsymbol{r}) = \frac{u_l(r)}{r}Y_{lm}(\hat{\boldsymbol{r}})\cdot\chi_{m_s}$$

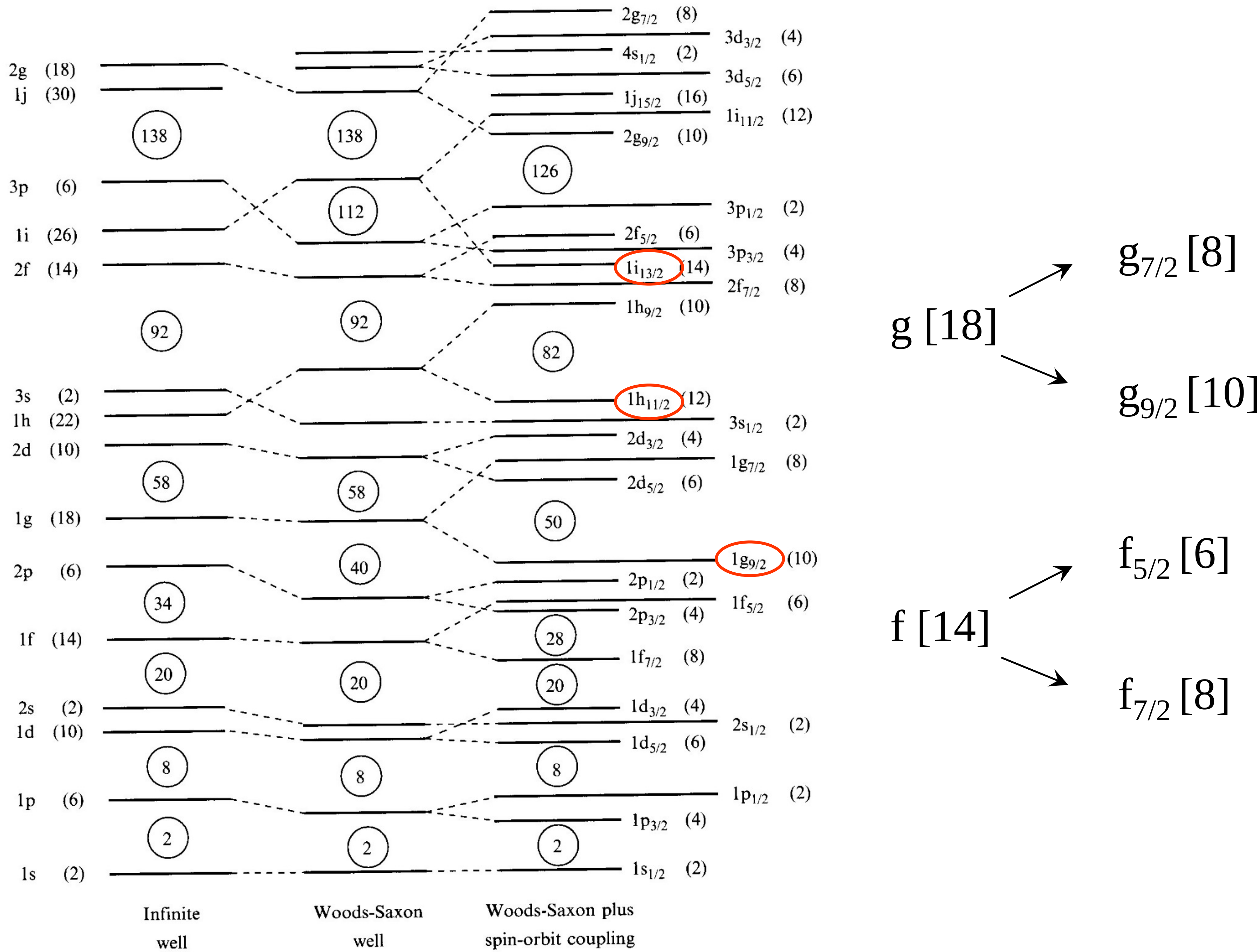
Spin-orbit interaction

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) + V_{ls}(r)\boldsymbol{l}\cdot\boldsymbol{s} - \epsilon\right]\psi(\boldsymbol{r}) = 0$$

(note) $\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s} \Longrightarrow \boldsymbol{l}\cdot\boldsymbol{s} = (\boldsymbol{j}^2 - \boldsymbol{l}^2 - \boldsymbol{s}^2)/2$

$$\boldsymbol{l}\cdot\boldsymbol{s} = \frac{l}{2}\ (j = l + 1/2), \quad -\frac{l+1}{2}\ (j = l - 1/2)$$




2g (18)
1j (30)
138
3p (6)
1i (26)
2f (14)
92
3s (2)
1h (22)
2d (10)
58
1g (18)
2p (6)
34
1f (14)
20
2s (2)
1d (10)
8
1p (6)
2
1s (2)
138
112
92
58
40
20
8
2
$2g_{7/2}$ (8)
$3d_{3/2}$ (4)
$4s_{1/2}$ (2)
$3d_{5/2}$ (6)
$1j_{15/2}$ (16)
$1i_{11/2}$ (12)
$2g_{9/2}$ (10)
126
$3p_{1/2}$ (2)
$2f_{5/2}$ (6)
$3p_{3/2}$ (4)
$1i_{13/2}$ (14)
$2f_{7/2}$ (8)
$1h_{9/2}$ (10)
82
$1h_{11/2}$ (12)
$3s_{1/2}$ (2)
$2d_{3/2}$ (4)
$1g_{7/2}$ (8)
$2d_{5/2}$ (6)
50
$1g_{9/2}$ (10)
$2p_{1/2}$ (2)
$1f_{5/2}$ (6)
$2p_{3/2}$ (4)
28
$1f_{7/2}$ (8)
20
$1d_{3/2}$ (4)
$2s_{1/2}$ (2)
$1d_{5/2}$ (6)
8
$1p_{1/2}$ (2)
$1p_{3/2}$ (4)
2
$1s_{1/2}$ (2)
Infinite well
Woods-Saxon well
Woods-Saxon plus spin-orbit coupling
g [18]
$g_{7/2}$ [8]
$g_{9/2}$ [10]
f [14]
$f_{5/2}$ [6]
$f_{7/2}$ [8]

Single particle spectra

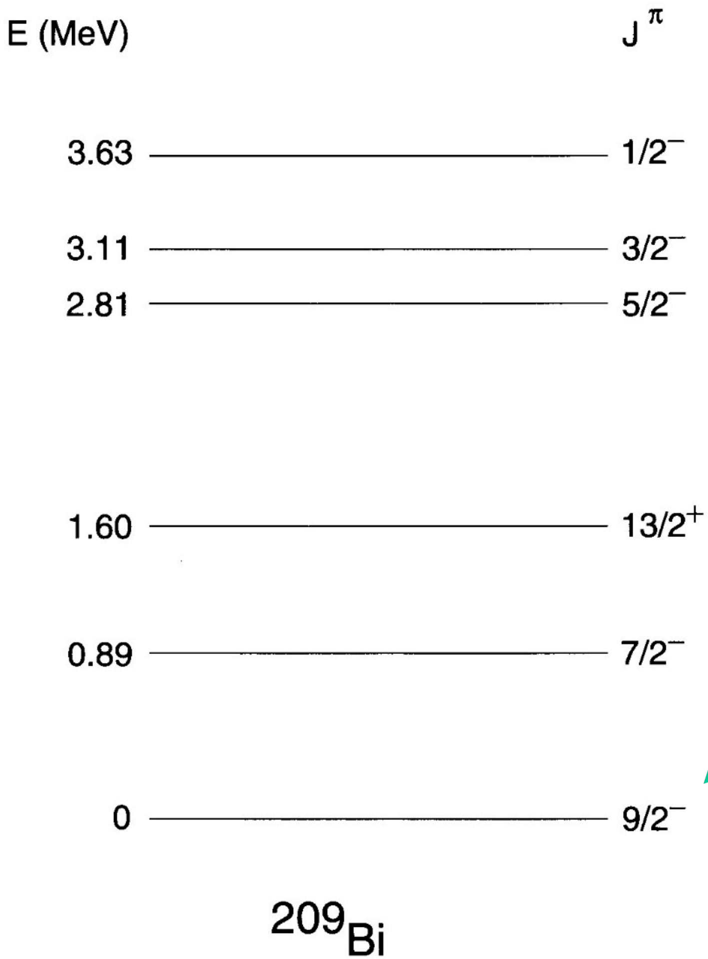
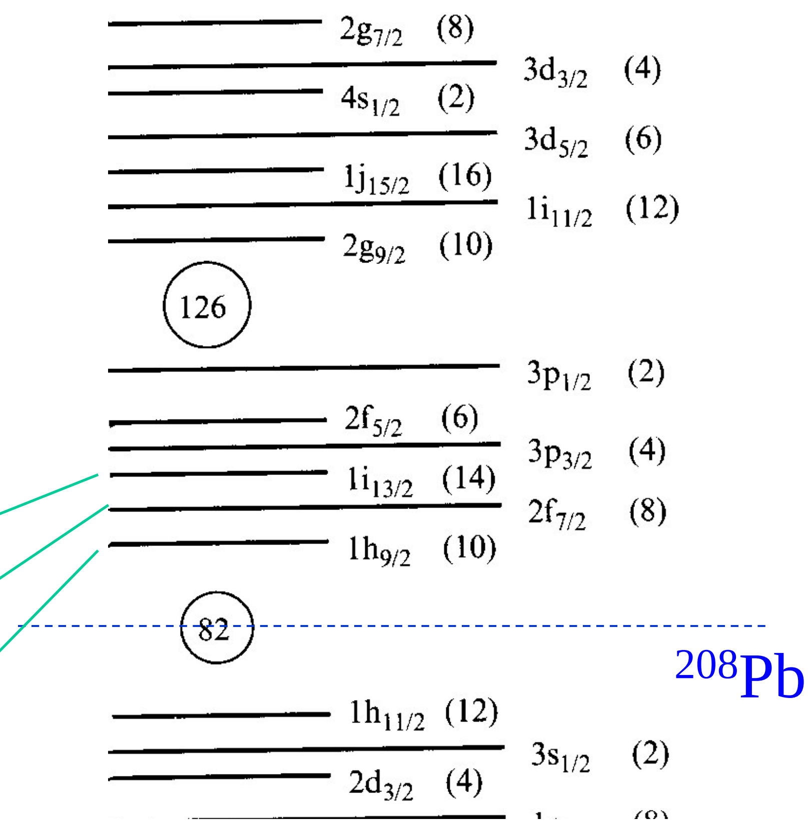


FIG. 3.6. Low-lying single-particle levels of ^{209}Bi.

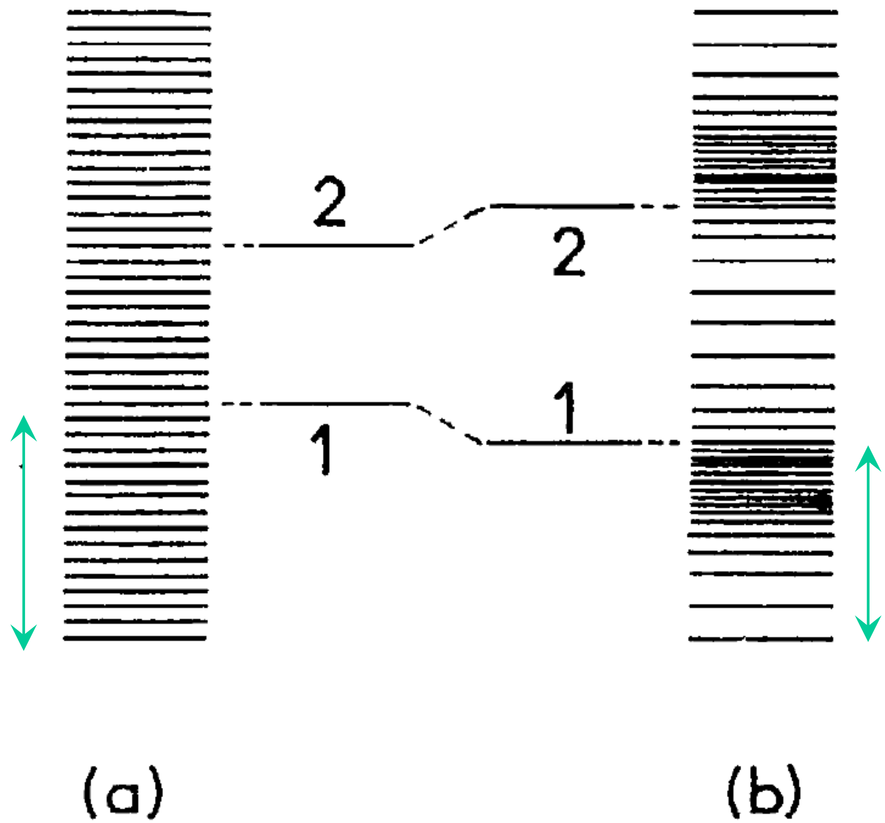


- How to construct V(r) microscopically?
- Does the independent particle picture really hold?

⇨ Later in this course

Why do closed-shell-nuclei become stable?

level density

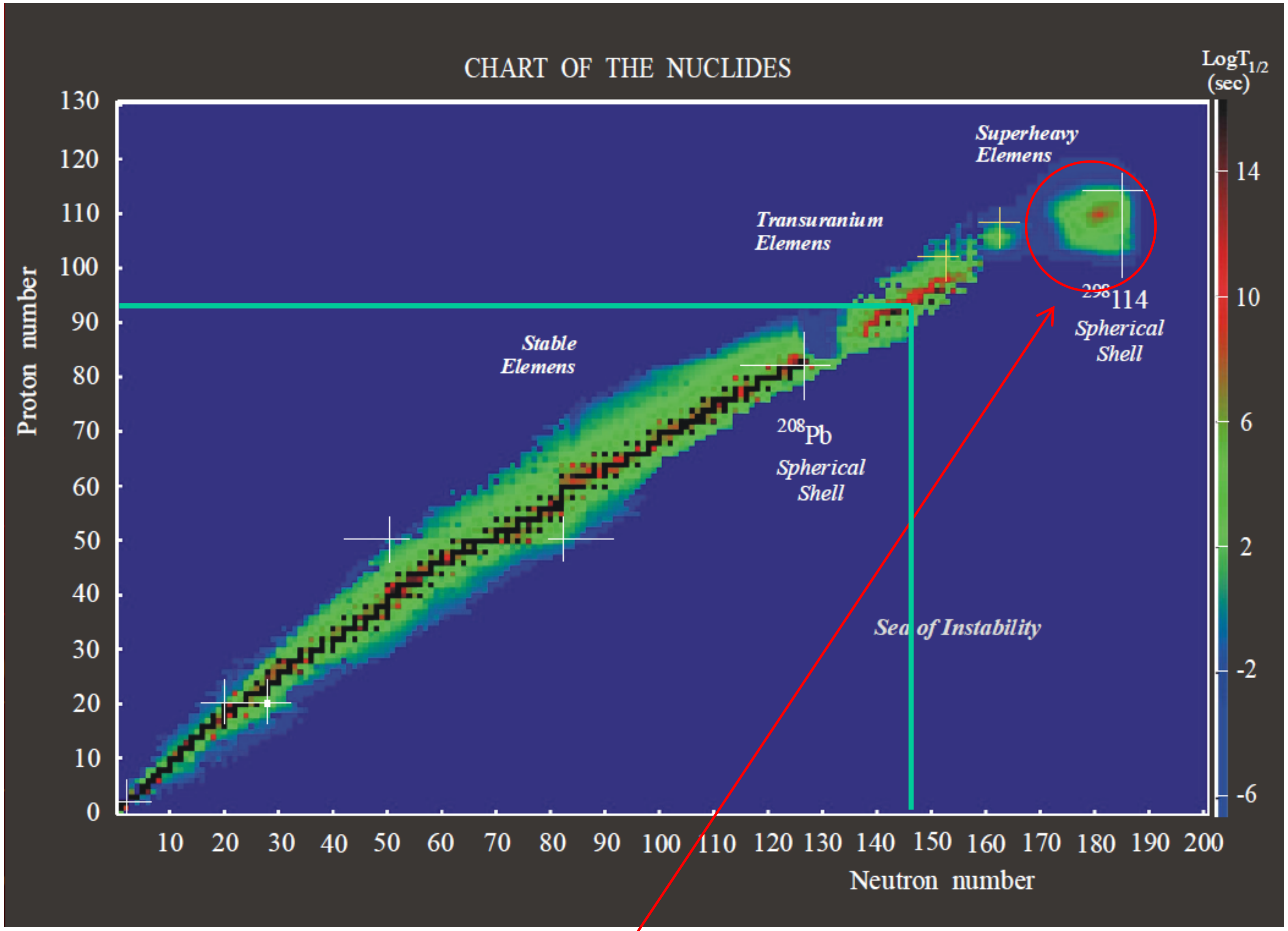


smaller total energy (more stable)

uniform

non-uniform

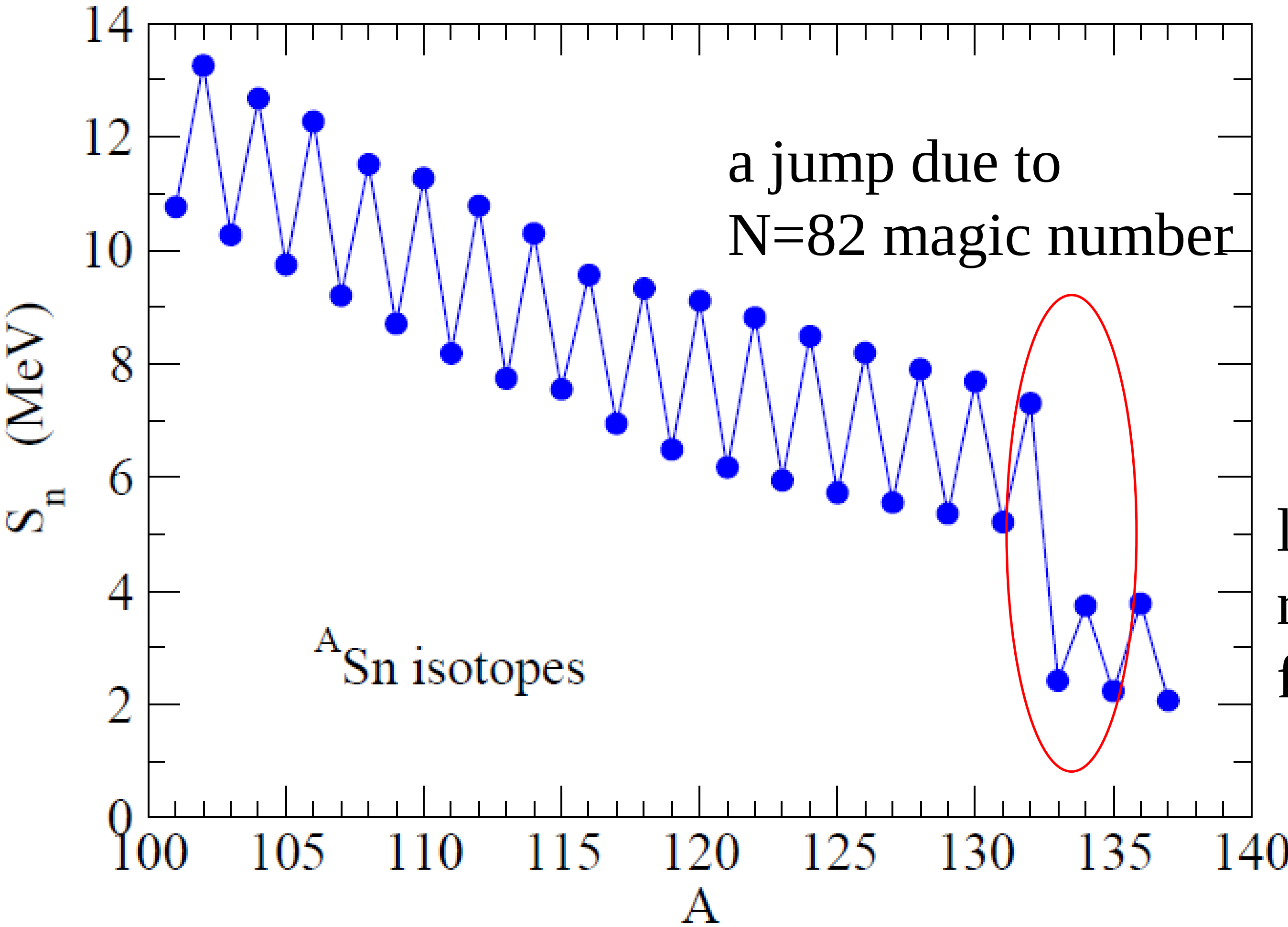
Theoretical prediction of island of stability



island of stability around Z=114, N=184

Yuri Oganessian

W.D. Myers and W.J. Swiatecki (1966), A. Sobiczewski et al. (1966)



less energy to
remove a neutron
from a higher orbit

1n separation energy: $S_n\,(A,Z) = B(A,Z) - B(A-1,Z)$

A lucky accident for the origin of life

Atomic magic numbers

electron #: 2, 10, 18, 36, 54, 86

inert gas：He, Ne, Ar, Kr, Xe, Rn

Nuclear magic numbers

proton # or neutron #

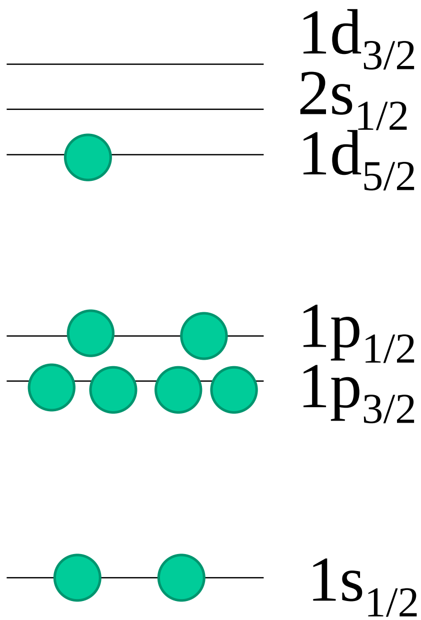
2, 8, 20, 28, 50, 82, 126

- e.g., ${}^{16}_{8}O_{8}$ (double magic)
- many oxygen nuclei: produced during nucleosynthesis
- oxygen: chemically active
- several complex chemical reactions, leading to the birth of life

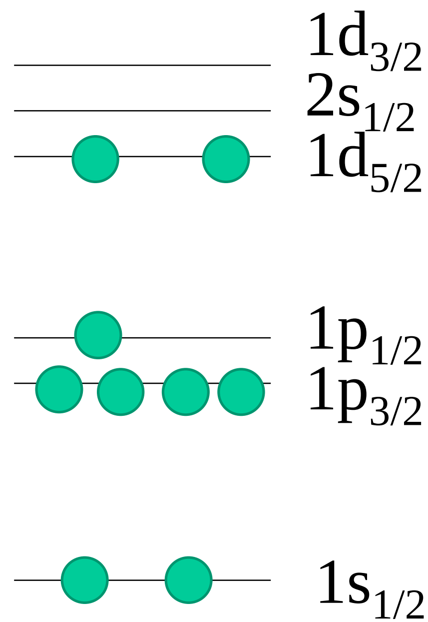
参考：望月優子　ビデオ「元素誕生の謎にせまる」　http://rarfaxp.riken.go.jp/~motizuki/contents/genso.html

single-j model

shell model



configuration 1



configuration 2

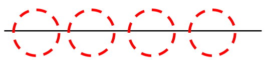
…… several others

angular momentum (spin) and parity for each configuration?

⟶ let us first investigate a single-j case single-j level: one level with an angular momentum j

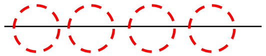
j

example: j = $p_{3/2}$



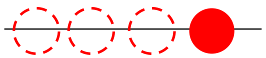
$p_{3/2}$

can accommodate 4 nucleons
(j_z= +3/2, +1/2, -1/2, -3/2)

 $p_{3/2}$ can accommodate 4 nucleons

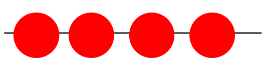
(j_z= +3/2, +1/2, -1/2, -3/2)

i) 1 nucleon

 $p_{3/2}$ $\Rightarrow$ $I^\pi = 3/2^-$

(there are 4 ways to occupy this level)

ii) 4 nucleons

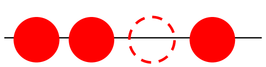
 $p_{3/2}$ $\Rightarrow$ $I^\pi = 0^+$

$$\boldsymbol{I} = \boldsymbol{j}_1 + \boldsymbol{j}_2 + \boldsymbol{j}_3 + \boldsymbol{j}_4$$

(there is only 1 way to occupy this level)

parity: (-1) x (-1) x (-1) x (-1) = +1

iii) 3 nucleons

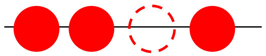
 $p_{3/2}$ $\Rightarrow$ $I^\pi = 3/2^-$

$$\boldsymbol{I} = \boldsymbol{j}_1 + \boldsymbol{j}_2 + \boldsymbol{j}_3$$

(there are 4 ways to make a hole)

parity: (-1) x (-1) x (-1) = -1

iii) 3 nucleons

 $p_{3/2}$

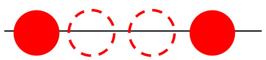
$I^\pi = 3/2^-$

$$\boldsymbol{I} = \boldsymbol{j}_1 + \boldsymbol{j}_2 + \boldsymbol{j}_3$$

(there are 4 ways to make a hole)

parity: (-1) x (-1) x (-1) = -1

iv) 2 nucleons

 $p_{3/2}$

$$\boldsymbol{I} = \boldsymbol{j}_1 + \boldsymbol{j}_2$$

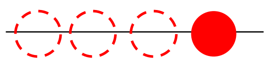
there are 4 x 3/2=6 ways to occupy this level with 2 nucleons.

$I^\pi = 0^+$ [1] or 2^+ [5]

$3/2 + 3/2 \longrightarrow I = 0, \cancel{1}, 2, \cancel{3}$

anti-symmetrization

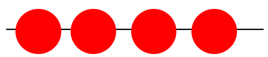
i) 1 nucleon



$p_{3/2}$ → $I^\pi = 3/2^-$

(there are 4 ways to occupy this level)

ii) 4 nucleons

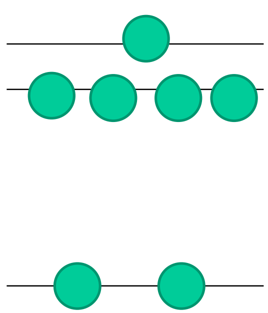


$p_{3/2}$ → $I^\pi = 0^+$

$$\boldsymbol{I} = \boldsymbol{j}_1 + \boldsymbol{j}_2 + \boldsymbol{j}_3 + \boldsymbol{j}_4$$

(there is only 1 way to occupy this level)

parity: (-1) x (-1) x (-1) x (-1) = +1



$1p_{1/2}$ → $I^\pi = 1/2^-$

$1p_{3/2}$ → $I^\pi = 0^+$

$1s_{1/2}$ → $I^\pi = 0^+$

in total,

$I^\pi = 1/2^-$

example: (main) shell model configurations for ${}^{11}_{5}B_6$

cf. ${}^{12}C(e,e'K^+){}^{12}_{\Lambda}B$ $(={}^{11}B+\Lambda)$

MeV

5.02 —— $3/2^-$

4.44 —— $5/2^-$

2.12 —— $1/2^-$

0 —— $3/2^-$

${}^{11}_{5}B_6$

cf. $^{12}C(e,e'K^+)^{12}_{\Lambda}B$ (=$^{11}B+\Lambda$)

PHYSICAL REVIEW C **90**, 034320 (2014)

Experiments with the High Resolution Kaon Spectrometer at JLab Hall C and the new spectroscopy of $^{12}_{\Lambda}B$ hypernuclei

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Ed V. Hungerford,[5] C. Jayalath,[1] M. Jones,[2] K. Johnston,[20] N. Kalantarians,[5] H. Kanda,[3] M. Kaneta,[3] F. Kato,[3] S. Kato,[21]
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owitz,[7] T. Maru llener,[22]
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Tsukada,[3] V. Tv s,[20]
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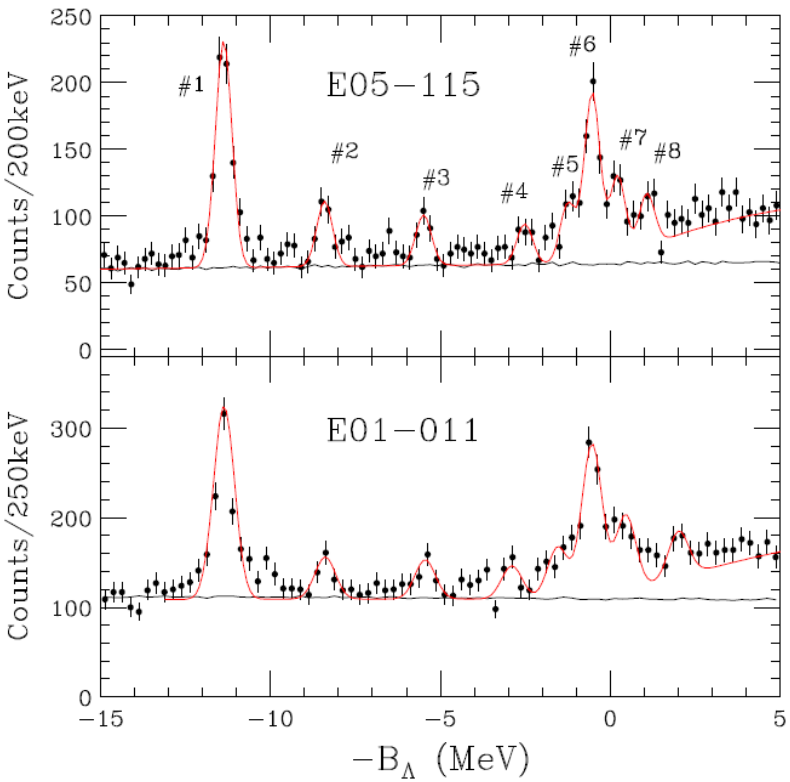
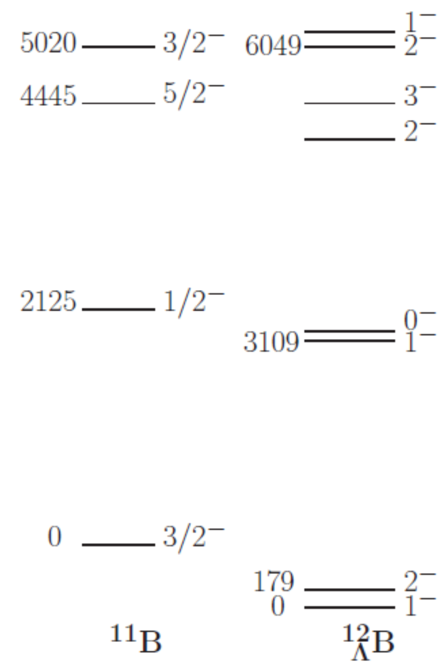
15 and E01-011

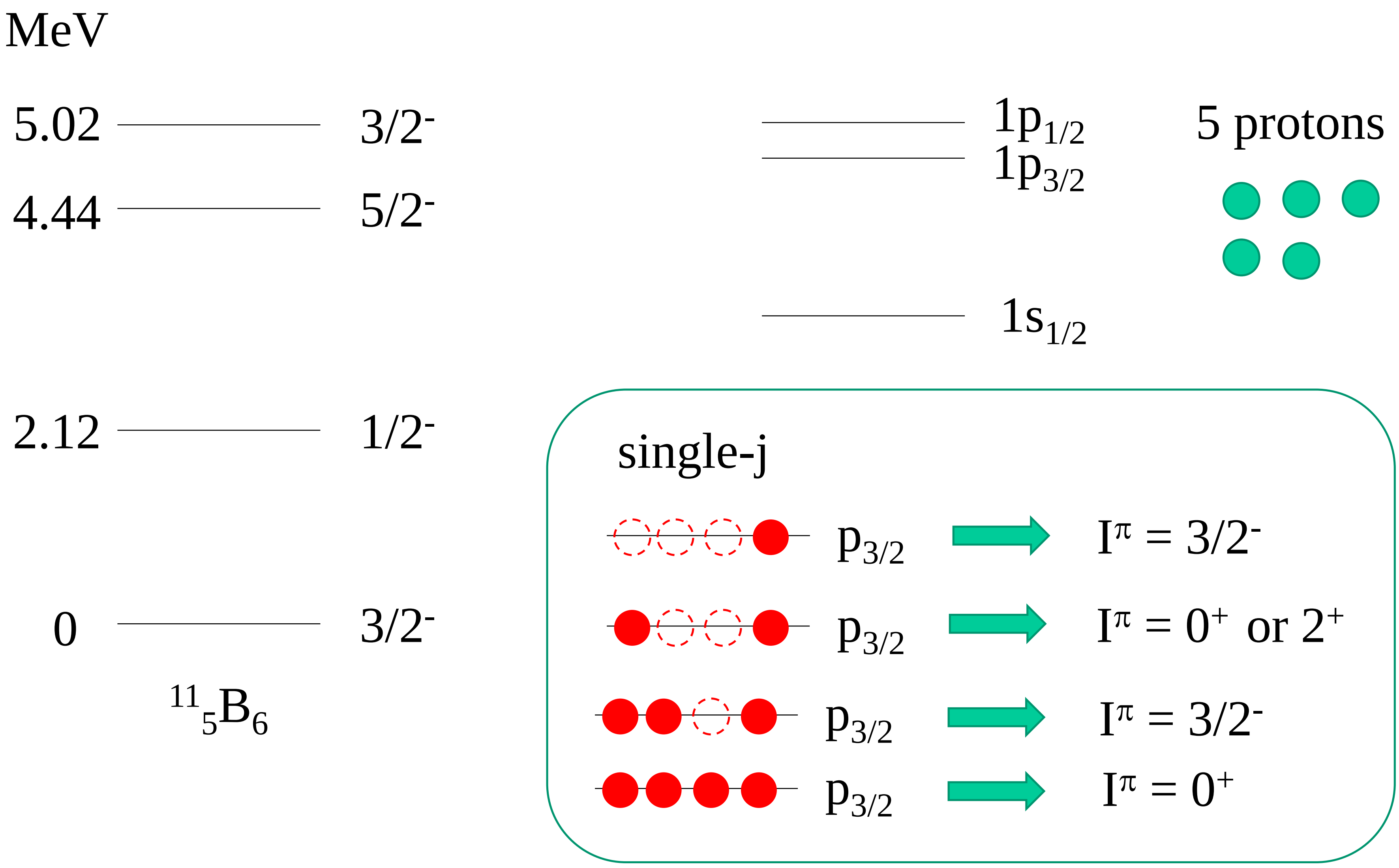




(b)

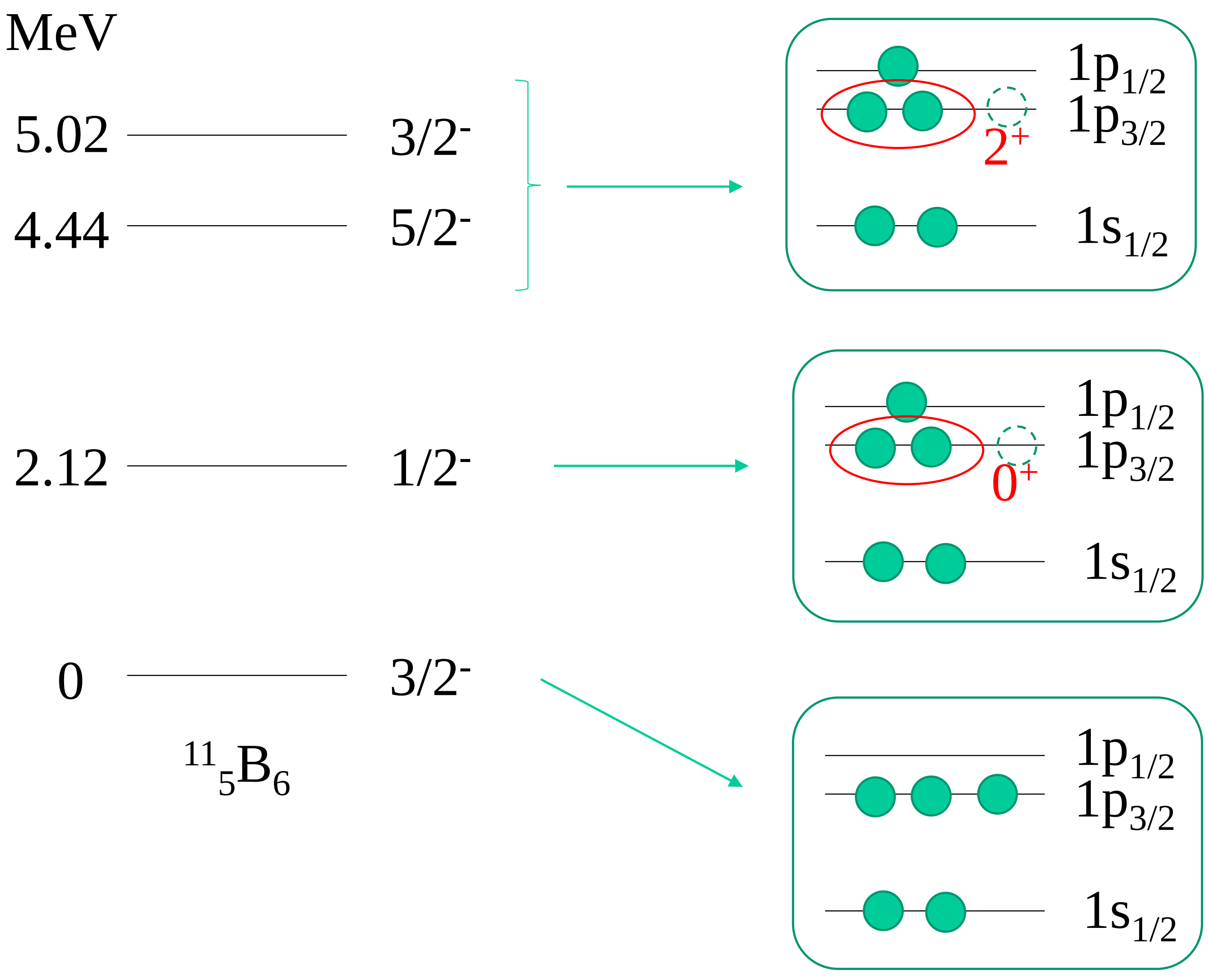
example: (main) shell model configurations for $^{11}_{5}B_{6}$

cf. $^{12}C(e,e'K^{+})^{12}_{\Lambda}B$ ($=^{11}B+\Lambda$)



example: (main) shell model configurations for ^{11}B

cf. $^{12}C(e,e'K^+)^{12}_{\Lambda}B$ (=$^{11}B+\Lambda$)



another example: (main) shell model configurations for ^{17}F

MeV	
4.64	$3/2^-$
3.10	$1/2^-$
0.495	$1/2^+$
0	$5/2^+$

$^{17}_{9}F_{8}$

another example: (main) shell model configurations for ^{17}F