Pairing Correlations

Mean-field approximation

independent particle motion in a potential well

\[ \psi(1, 2, \cdots, A) = A[\psi_1(1)\psi_2(2)\cdots\psi_A(A)] \]

\[ = \frac{1}{\sqrt{A!}} \begin{vmatrix} \psi_1(1) & \psi_2(1) & \cdots & \psi_A(1) \\ \psi_1(2) & \psi_2(2) & \cdots & \psi_A(2) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(A) & \psi_2(A) & \cdots & \psi_A(A) \end{vmatrix} \]

Slater determinant: antisymmetrization due to the Pauli principle

nn interaction: only through a mean-field potential
$^{16}\text{O}$

$\frac{3}{2}^+$  \hspace{1cm} 5.08\text{ MeV}

$\frac{1}{2}^+$  \hspace{1cm} 0.87\text{ MeV}

$\frac{5}{2}^+$  \hspace{1cm} 0

$^{17}_{8}\text{O}_9$
Pairing correlation

What if two neutrons are put outside the core nucleus?

What is the influence of the interaction between the two neutrons?
What is the influence of the interaction between the two neutrons?

**Mean-field theory**

treat the interaction among particles only on average

the pure mean-field picture

→ the interaction between the two neutrons: only through the mean-field potential, (the two neutrons: uncorrelated).
\(1/2^+ \quad 0.87 \text{ MeV}\)

\(5/2^+ \quad 0\)

\^{17}_8\text{O}_9

\^{16}_8\text{O}

\(\text{at least 6 levels below 2 MeV (?)}\)
pure mean-field approximation:

\[
\begin{align*}
0^+ & \quad 1.74 \text{ MeV} \\
2^+, 3^+ & \quad 0.78 \text{ MeV} \\
0^+, 2^+, 4^+ & \quad 0 \\
^{18}_8\text{O}_{10} & \quad \text{6 levels} \\
\end{align*}
\]

in reality:

\[
\begin{align*}
2^+ & \quad 1.98 \text{ MeV} \\
0^+ & \quad 0 \\
^{18}_8\text{O}_{10} & \quad \text{only two levels!}
\end{align*}
\]
what is going on?

\[ H = \sum_i T_i + \sum_{i<j} v_{ij} \rightarrow H = \sum_i (T_i + V_i) + \sum_{i<j} v_{ij} - \sum_i V_i \]

deviation from the average (residual interaction)

Can the residual interaction be neglected completely?

→ “no” for open-shell nuclei (pairing correlation)
Pairing correlation

\[ H = \sum_{i=1}^{A} \left( -\frac{\hbar^2}{2m} \nabla_i^2 + V_{HF}(i) \right) + \frac{1}{2} \sum_{i,j}^{A} v(r_i, r_j) - \sum_{i}^{A} V_{HF}(i) \]  
\[ v_{\text{res}}(r, r') \]

A delta function interaction for a residual interaction:  
(an extremely short range interaction)

\[ v_{\text{res}}(r, r') \sim -g \, \delta(r - r') \]

\[ = -g \frac{\delta(r - r')}{r_{rr'}} \sum_{\lambda \mu} Y_{\lambda \mu}^* (\hat{r}) Y_{\lambda \mu} (\hat{r}') \]

Estimate the effect of \( v_{\text{res}} \) using the perturbation theory:

unperturbative wave function:

\[ l \quad \text{two neutrons in a angular momentum } l \text{ state} \]

with the total angular momentum \( L \)

\[ |(ll)LM \rangle = \sum_{m,m'} \langle llm m'|LM \rangle \psi_{lm}(r) \psi_{lm'}(r') \]
Pairing correlations

\[ v_{\text{res}}(r, r') \sim -g \delta(r - r') \]
\[ = -g \frac{\delta(r - r')}{r r'} \sum_{\lambda \mu} Y_{\lambda \mu}^*(\hat{r}) Y_{\lambda \mu}(\hat{r}') \]

\[ |(ll)LM\rangle = \sum_{m, m'} \langle lmlm' |LM\rangle \psi_{lm}(r) \psi_{lm'}(r') \]

The energy change due to the residual interaction:

\[ \Delta E_L = \langle (ll)LM | v_{\text{res}} | (ll)LM \rangle \]
\[ = -g I_r^{(l)} \frac{(2l + 1)^2}{4\pi} \begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}^2 \]

\[ \psi_{lm}(r) = R_l(r) Y_{lm}(\hat{r}) \quad I_r^{(l)} = \int_0^\infty r^2 dr \left( R_l(r) \right)^4 \]
\[ \Delta E_L = -g I_r^{(l)} \frac{(2l + 1)^2}{4\pi} \left( \begin{array}{ccc} l & l & L \\ 0 & 0 & 0 \end{array} \right)^2 \equiv -g I_r^{(l)} \frac{A(ll; L)}{4\pi} \]

<table>
<thead>
<tr>
<th>(A(ll; L))</th>
<th>(L=0)</th>
<th>(L=2)</th>
<th>(L=4)</th>
<th>(L=6)</th>
<th>(L=8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l = 2)</td>
<td>5.00</td>
<td>1.43</td>
<td>1.43</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>(l = 3)</td>
<td>7.00</td>
<td>1.87</td>
<td>1.27</td>
<td>1.63</td>
<td>---</td>
</tr>
<tr>
<td>(l = 4)</td>
<td>9.00</td>
<td>2.34</td>
<td>1.46</td>
<td>1.26</td>
<td>1.81</td>
</tr>
</tbody>
</table>

\(0^+, 2^+, 4^+, 6^+, \ldots\)

\[ \begin{align*}
0^+ & \rightarrow \begin{array}{c}
4^+ \\
6^+ \\
2^+ \\
0^+
\end{array}
\end{align*} \]

without residual interaction

with residual interaction
Simple interpretation:

The spatial overlap is the largest for an $L = 0$ pair.

“The Pairing Correlation”

(note) The $L=2l$ pair is unfavoured due to the Pauli principle.

(note)

$$\psi(l^2; L = 0) = \sum_{\mu} \langle l\mu l - \mu|L = 0, 0 \rangle Y_{l\mu}(\hat{r}_1)Y_{l-\mu}(\hat{r}_2) = Y_{l0}(\theta_{12})/\sqrt{4\pi}$$
“Pairing Correlation”

The ground state spin of nuclei

- Even-even nuclei: $0^+$
- Even-odd nuclei: the spin of the valence particle
pure mean-field approximation:

\[ \begin{align*}
0^+ & \quad 1.74 \text{ MeV} \\
2^+, 3^+ & \quad 0.78 \text{ MeV} \\
0^+, 2^+, 4^+ & \quad 0 \\
\end{align*} \]
pure mean-field approximation:

\[ 0^+ \quad 1.74 \text{ MeV} \]
\[ 2^+, 3^+ \quad 0.78 \text{ MeV} \]

\[ ^{18}_8 \text{O}_{10} \]
Pairing energy

even-odd staggering

A larger energy required to remove one neutron from even number than from odd number

“pair correlation”

even-even nuclei

even-odd nuclei

\[ S_n (A,Z) = B(A,Z) - B(A-1,Z) \]

\( A^{\text{Sn}} \) isotopes
Wave functions:

Each orbit is occupied only partially.

cf. BCS theory (super fluidity/super conductivity)
Fifty Years of Nuclear BCS
Pairing in Finite Systems

Ricardo A Broglia
Vladimir Zelevinsky

World Scientific
Role of residual interaction

\[ H = \sum_i T_i + \sum_{i<j} v_{ij} \rightarrow H = \sum_i (T_i + V_i) + \sum_{i<j} v_{ij} - \sum_i V_i \]

residual interaction (pairing)

open shell nuclei \( \rightarrow \) superfluidity

weakly bound nuclei
Borromean nucleus

residual interaction $\rightarrow$ attractive

Structure of Borromean nuclei

✓ non-trivial due to many-body correlations
✓ has attracted lots of attention
Borromean nuclei

Another typical example: $^6\text{He}$
What is “Borromean”? 

Even though three rings are tied together, two rings can be separated once any of three is removed.

“Borromean rings”
What is “Borromean”? 

Borromean islands (northern Italy, in Lake Maggiore) near Milano

Crest of Borromeano Family (13th century)
① Mean-field approximation for a mean-field potential (first, an average behavior)

② Next, an occupation probability for each level based on the variational principle including the residual interaction

\[
|BCS\rangle = \prod_{k>0} \left( u_k + v_k a_k^\dagger a_k^\dagger \right) |0\rangle
\]

\[
a_k^\dagger = a_{jlm}^\dagger, \quad a_k^\dagger = (-)^{l+j-m} a_{jl-m}^\dagger
\]

\[
\langle BCS|a_k^\dagger a_k|BCS\rangle = |v_k|^2 \quad \text{occupation probability}
\]
For the Hamiltonian:

\[ H = \sum_\nu \epsilon_k (a_k^\dagger a_k + a_k^\dagger a_{-k}^\dagger) - G \left( \sum_{k>0} a_k^\dagger a_{-k}^\dagger \right) \left( \sum_{k>0} a_{-k} a_k \right) \]

\[ u_\nu^2 = \frac{1}{2} \left( 1 + \frac{\epsilon_\nu - \lambda}{E_k} \right) \]

\[ v_\nu^2 = \frac{1}{2} \left( 1 - \frac{\epsilon_\nu - \lambda}{E_k} \right) \]

\[ E_k = \sqrt{(\epsilon_k - \lambda)^2 + \Delta^2} \]

\( \lambda \): chemical potential (Fermi energy)

\[ \Delta = G \langle BCS | \sum_{k>0} a_k^\dagger a_{-k}^\dagger | BCS \rangle = G \sum_{\nu>0} u_\nu v_\nu \]

\[ = \frac{G}{2} \sum_{\nu>0} \frac{\Delta}{E_\nu} \]

pairing gap
i) Trivial solution: always exists

\[ \Delta = 0 \]

\[ v_\nu^2 = 1 \ (\epsilon_\nu \leq \lambda) \]

\[ = 0 \ (\epsilon_\nu > \lambda) \]

\[ |\psi\rangle = \prod_{\nu > 0} a_\nu^\dagger a_\nu^\dagger |0\rangle \]

\[ G \text{ a/o } N \rightarrow \text{ large} \]

ii) Superfluid solution

\[ \Delta \neq 0 \]

\[ v_\nu^2 < 1 \]

\[ |BCS\rangle = \prod_{\nu > 0} (u_\nu + v_\nu a_\nu^\dagger a_\nu^\dagger) |0\rangle \]

the number fluctuation

Normal-Superfulid phase transition
Quasi-particle excitations

\[ H \sim E_{BCS} + \sum_k E_k \alpha_k^\dagger \alpha_k \]
\[ \alpha_k |BCS\rangle = 0 \]

\[ E_k = \sqrt{(\epsilon_k - \lambda)^2 + \Delta^2} \]

\[ \alpha_{\nu}^\dagger = u_{\nu} a_{\nu}^\dagger - v_{\nu} a_{\nu}, \quad \alpha_{\nu}^\dagger = u_{\nu} a_{\nu}^\dagger + v_{\nu} a_{\nu} \]

(Bogoliubov transformation)

(note) \[ E_k = \sqrt{(\epsilon_k - \lambda)^2 + \Delta^2} \geq \Delta \] (energy gap)

✓ a nucleus with N+1 nucleons: \[ \alpha_{\nu}^\dagger |BCS\rangle \]

✓ excited states of the same nucleus: \[ \alpha_{\nu}^\dagger \alpha_{\nu'}^\dagger |BCS\rangle \]

(note) \[ \alpha^\dagger \alpha^\dagger \sim a^\dagger a^\dagger + a^\dagger a + a a^\dagger + a a \]
Figure 6.1. Excitation spectra of the $^{50}\text{Sn}$ isotopes.
Even-odd mass difference and pairing gap

\[
E(N + 2, Z) = E(N, Z) + 2\lambda \\
E(N + 1, Z) = E(N, Z) + \lambda + \Delta
\]

\[\Delta_n \sim \frac{[E(N + 2, Z) - 2E(N + 1, Z) + E(N, Z)]}{2}\]
Even-odd mass difference and pairing gap

\[ E(N + 2, Z) = E(N, Z) + 2\lambda \]
\[ E(N + 1, Z) = E(N, Z) + \lambda + \Delta \]

(note) \( \lambda < 0 \)

\[ -\Delta_n \sim [E(N + 2, Z) - 2E(N + 1, Z) + E(N, Z)]/2 \]
Even-odd mass difference and pairing gap

\[ B_{\text{pair}} = \Delta \quad (\text{for even} - \text{even}) \quad E(N + 2, Z) = E(N, Z) + 2\lambda \]
\[ = 0 \quad (\text{for even} - \text{odd}) \quad E(N + 1, Z) = E(N, Z) + \lambda + \Delta \]
\[ = -\Delta \quad (\text{for odd} - \text{odd}) \]

\[-\Delta_n \sim [E(N + 2, Z) - 2E(N + 1, Z) + E(N, Z)]/2\]

\[ \Delta \sim 12/\sqrt{A} \quad (\text{MeV}) \]

Bohr-Mottelson ('69)
Hartree-Fock-Bogoliubov (HFB) Theory

HF+BCS method: two-step procedure
   (first MF potential, then occupation probabilities)

\[ \psi_k(r), u_k, v_k \]

improvement: MF and occ. prob. at the same time

Hartree-Fock-Bogoliubov (HFB) theory:

wave function + occupation probabilities

\[ U_k(r), V_k(r) \]
Application of the HFB method

Density of $^{110}$Zr (SHFB-SLy4)

A. Blazkiewicz et al., PRC71(’05)054231

Systematics of $\beta_2$ and $S_{2n}$

M.V. Stoitsov et al., PRC68(’03)054312
potential energy surface for fission process

A. Staszczak, A. Baran, J. Dobaczewski, and W. Nazarewicz, PRC80 (‘09) 014309