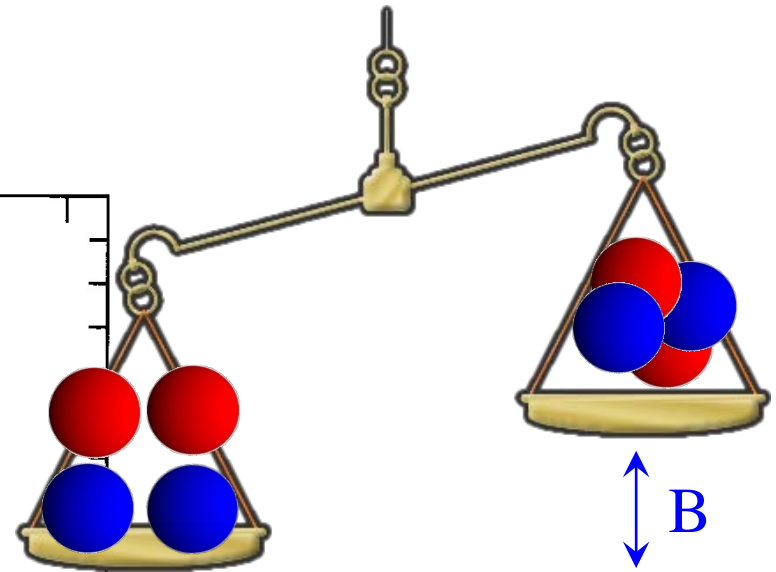
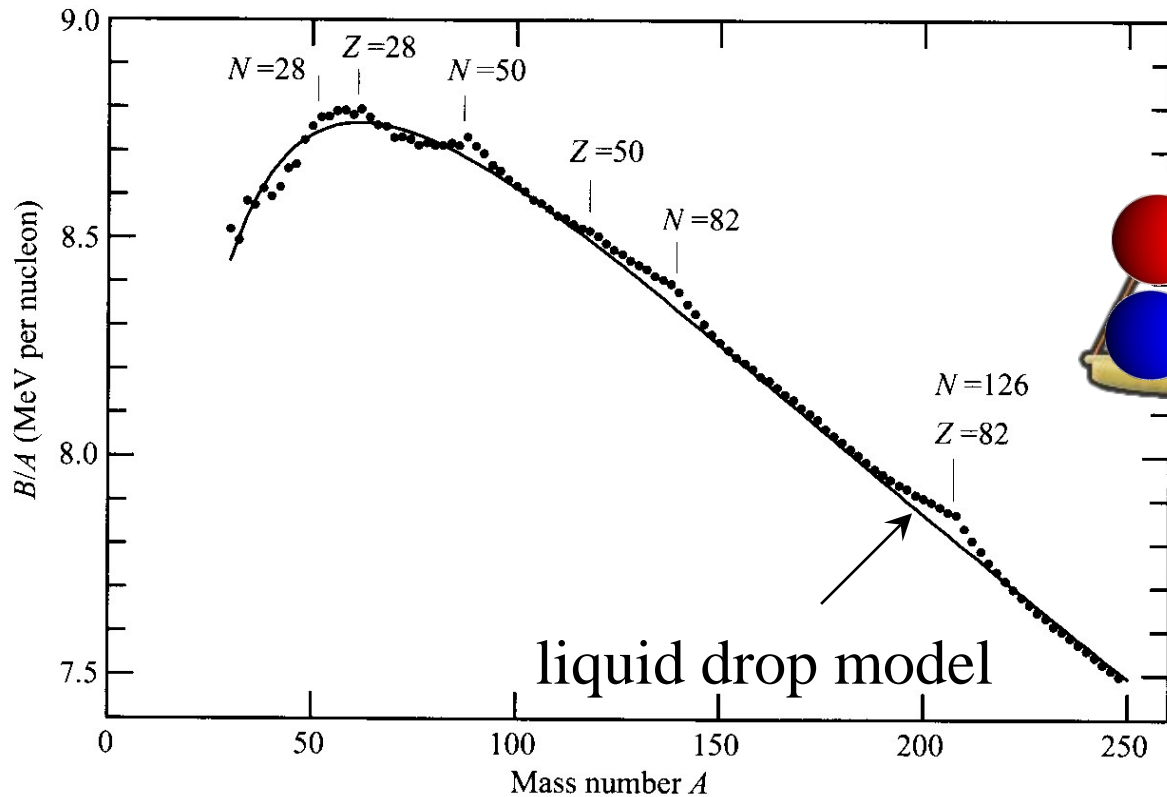


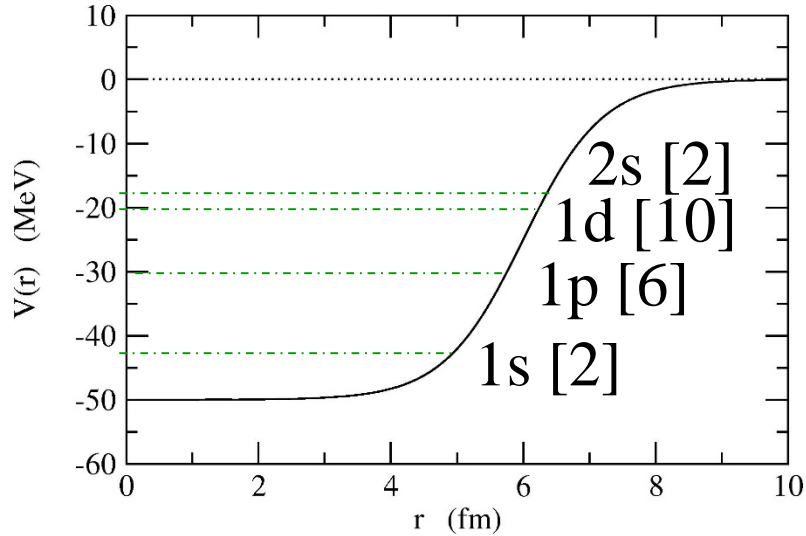
Nuclear magic numbers



Extra binding for N or $Z = 2, 8, 20, 28, 50, 82, 126$ (magic numbers)

Extra binding for N or $Z = 2, 8, 20, 28, 50, 82, 126$ (magic numbers)

An interpretation: independent particle motion in a potential well



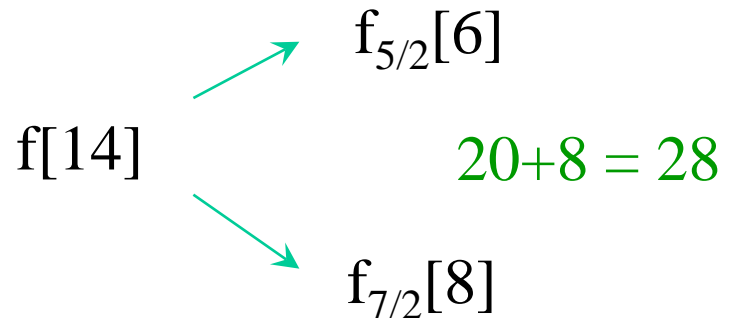
$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(r) - \epsilon \right] \psi(\mathbf{r}) = 0$$

$$\psi(\mathbf{r}) = \frac{u_l(r)}{r} Y_{lm}(\hat{\mathbf{r}}) \cdot \chi_{m_s}$$

degeneracy: $2 \cdot (2l+1)$

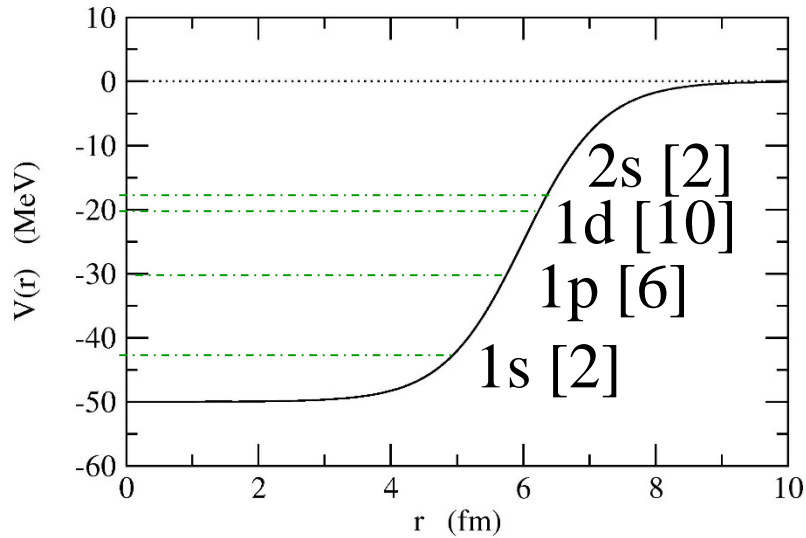
spin-orbit interaction

f[14]	34
s[2],d[10]	20
p[6]	8
s[2]	2



Extra binding for N or $Z = 2, 8, 20, 28, 50, 82, 126$ (magic numbers)

An interpretation: independent particle motion in a potential well



+ spin-orbit interaction

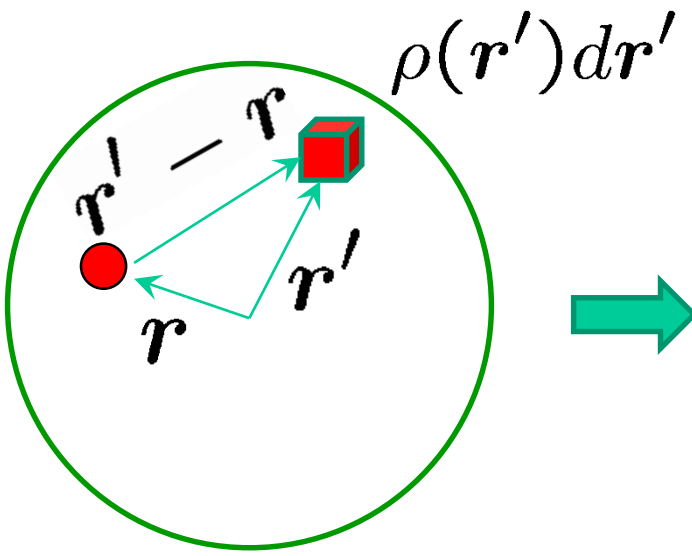
Today: how to construct the potential well?

Mean-field (Hartree-Fock) Theory

nucleon-nucleon interaction



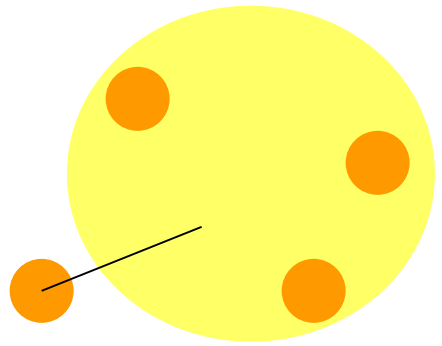
interaction for a nucleon inside a nucleus:



$\Rightarrow v(r' - r) \cdot \rho(r')dr'$

the number of nucleon at r'

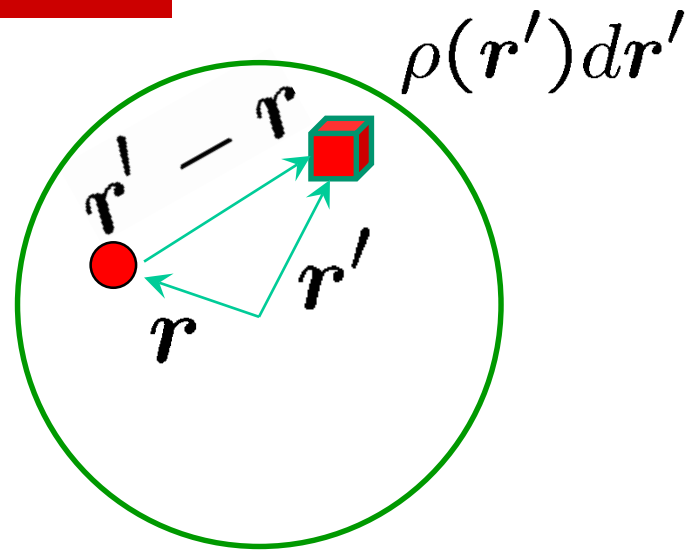
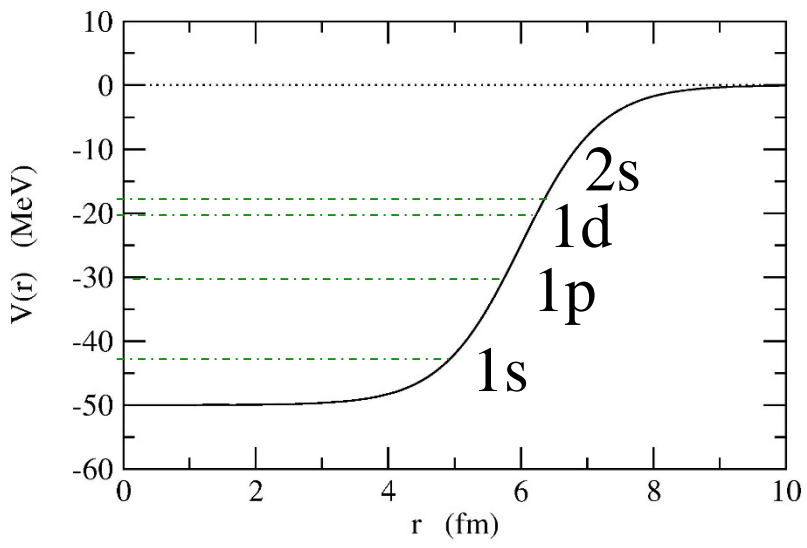
平均場



naively speaking,

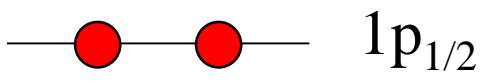
$$V(r) \sim \int v(r - r')\rho(r')dr'$$

Mean-field (Hartree-Fock) Theory

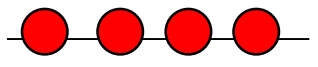


naively speaking,

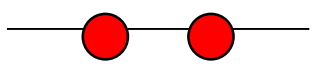
$$V(r) \sim \int v(r - r') \rho(r') dr'$$



$1p_{1/2}$



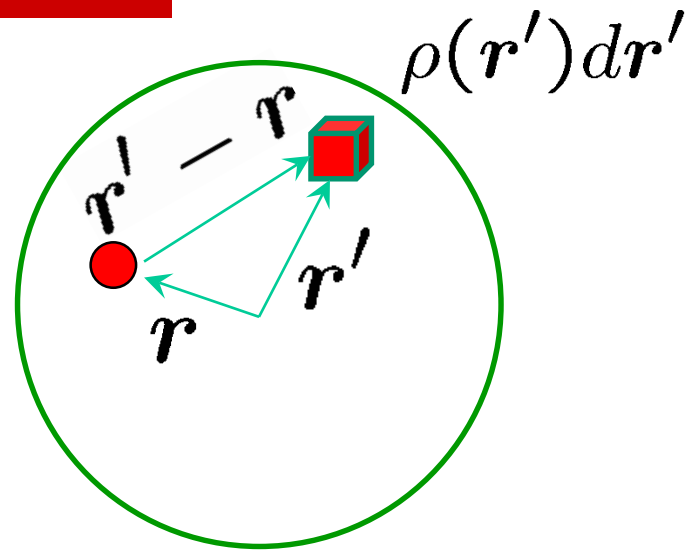
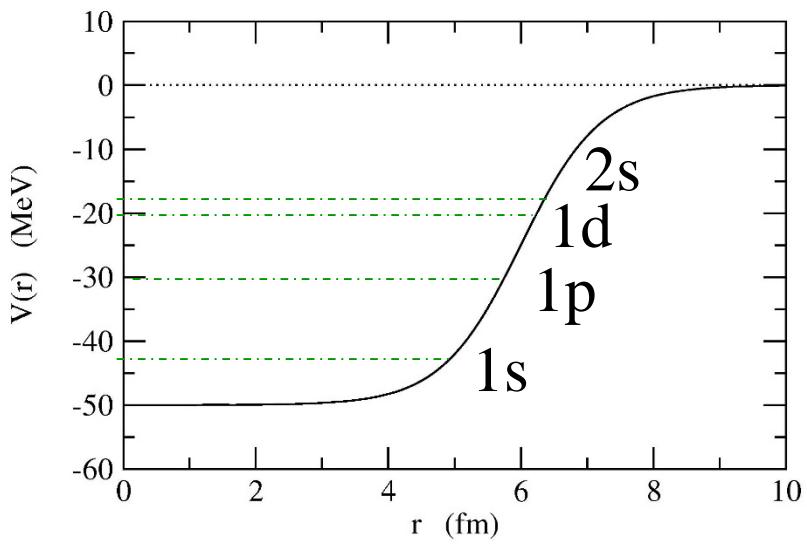
$1p_{3/2}$



$1s_{1/2}$

shell model

Mean-field (Hartree-Fock) Theory

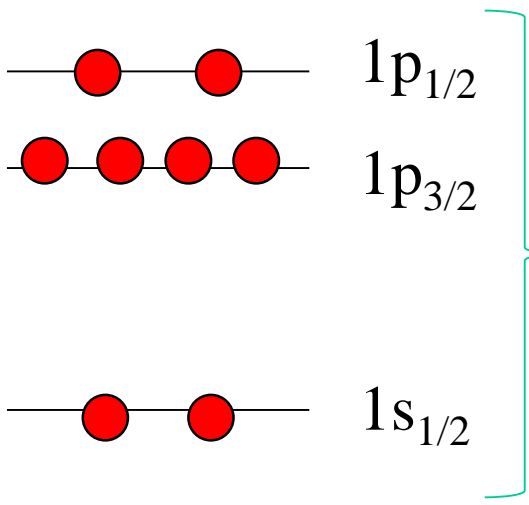


naively speaking,

$$V(\mathbf{r}) \sim \int v(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') d\mathbf{r}'$$

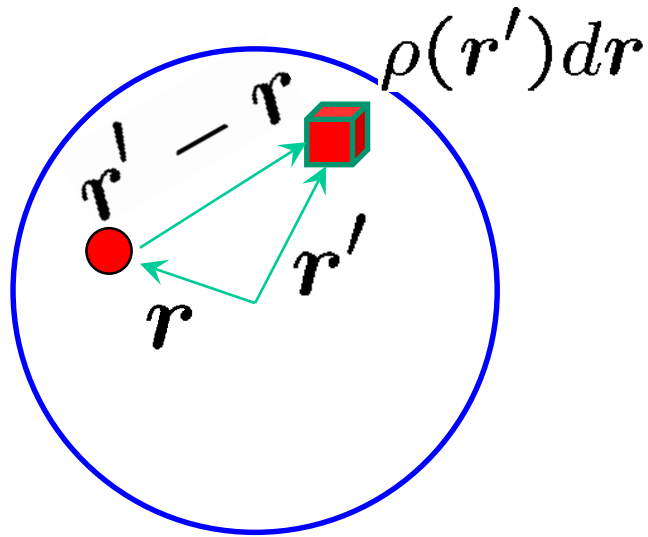
independent motion

$$\rho(\mathbf{r}) = \sum_i |\psi_i(\mathbf{r})|^2$$



shell model

Mean-field (Hartree-Fock) Theory



naively speaking,

$$V(\mathbf{r}) \sim \int v(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') d\mathbf{r}'$$

$$\rho(\mathbf{r}) = \sum_i |\psi_i(\mathbf{r})|^2$$

$$\begin{aligned} 0 &= \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) - \epsilon_i \right] \psi_i(\mathbf{r}) \\ &= \left[-\frac{\hbar^2}{2m} \nabla^2 + \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j |\psi_j(\mathbf{r}')|^2 \right) d\mathbf{r}' - \epsilon_i \right] \psi_i(\mathbf{r}) \end{aligned}$$

the potential depends on the solutions

Mean-field (Hartree-Fock) Theory

$$\begin{aligned} 0 &= \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) - \epsilon_i \right] \psi_i(\mathbf{r}) \\ &= \left[-\frac{\hbar^2}{2m} \nabla^2 + \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j |\psi_j(\mathbf{r}')|^2 \right) d\mathbf{r}' - \epsilon_i \right] \psi_i(\mathbf{r}) \end{aligned}$$

the potential depends on the solutions

→ **self-consistent solutions**

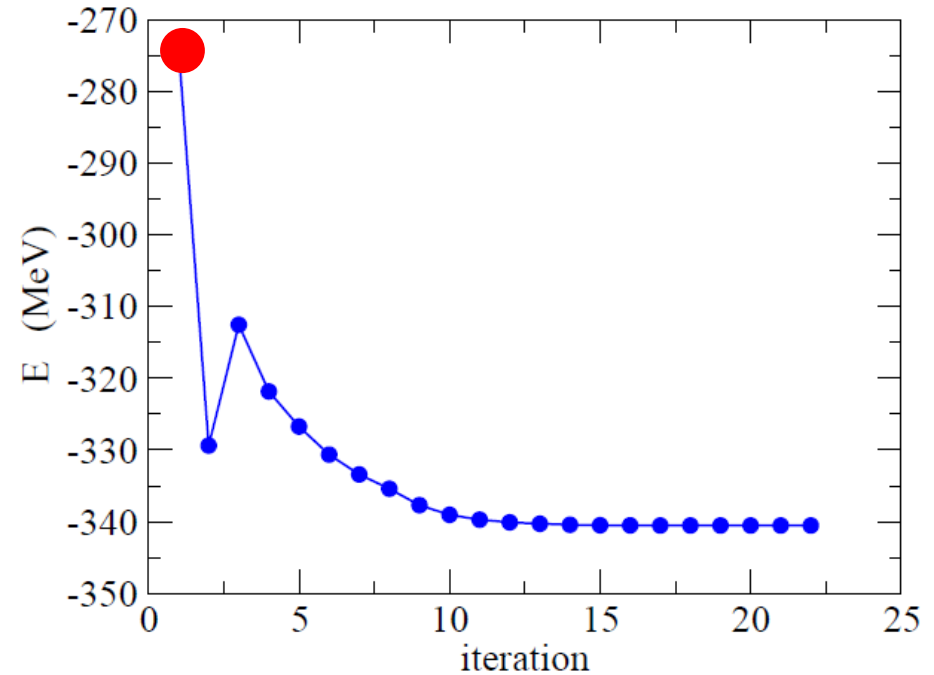
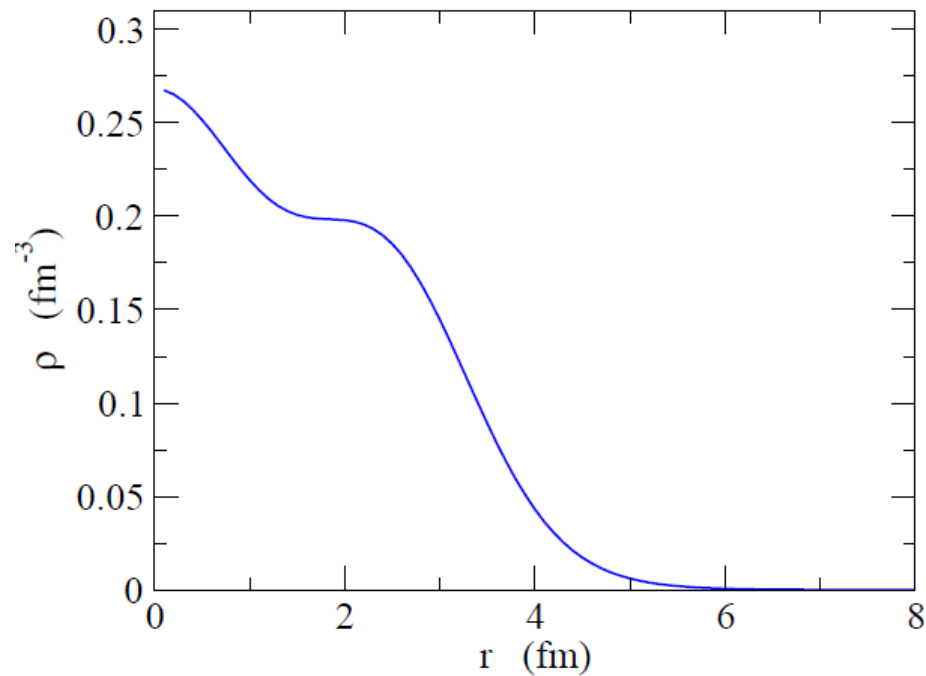
$$\text{Iteration: } \{\psi_i\} \rightarrow \rho \rightarrow V \rightarrow \{\psi_i\} \rightarrow \dots$$

$$\rho(\mathbf{r}) = \sum_i |\psi_i(\mathbf{r})|^2, \quad V(\mathbf{r}) \sim \int v(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') d\mathbf{r}'$$

repeat until the first and the last wave functions are the same.

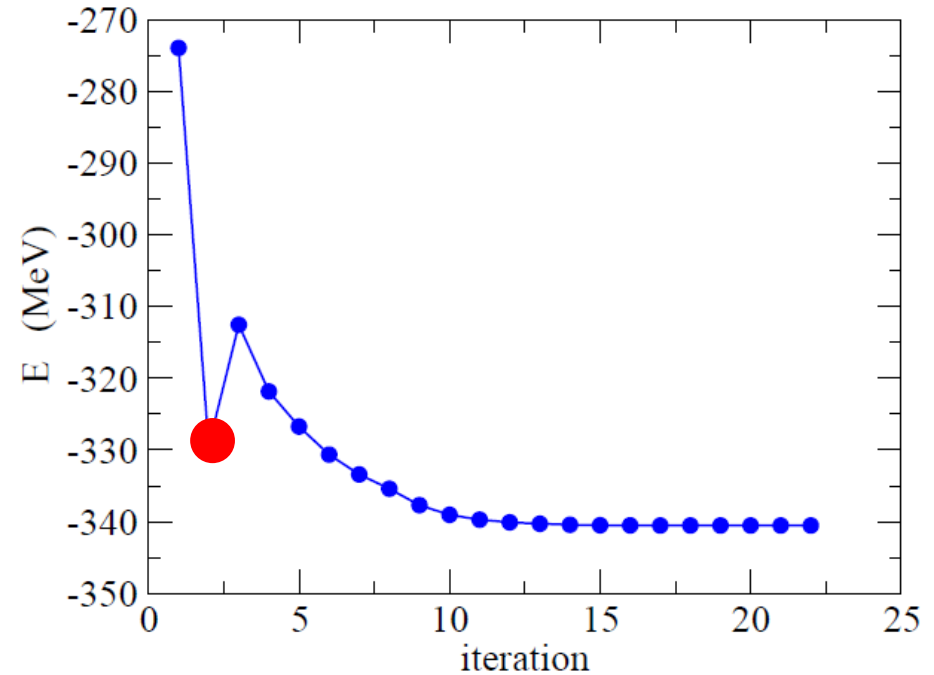
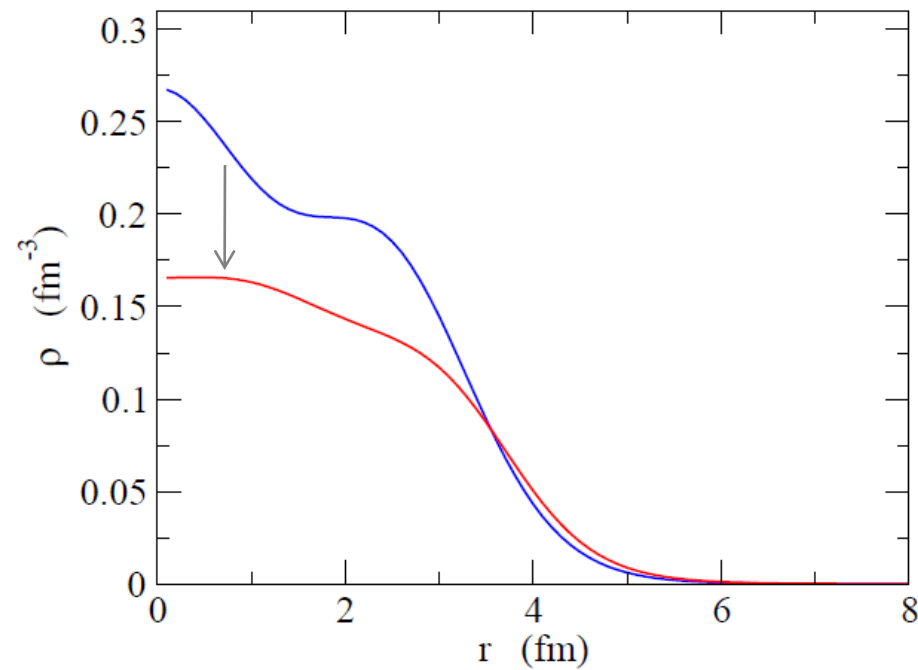
“self-consistent solutions”

Skyrme-Hartree-Fock calculations for ^{40}Ca



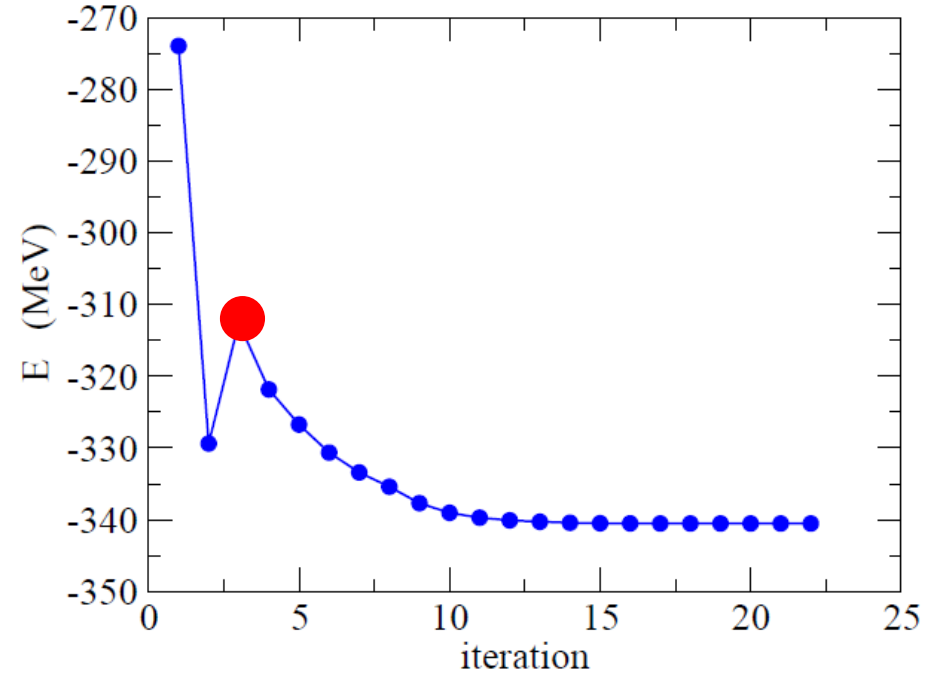
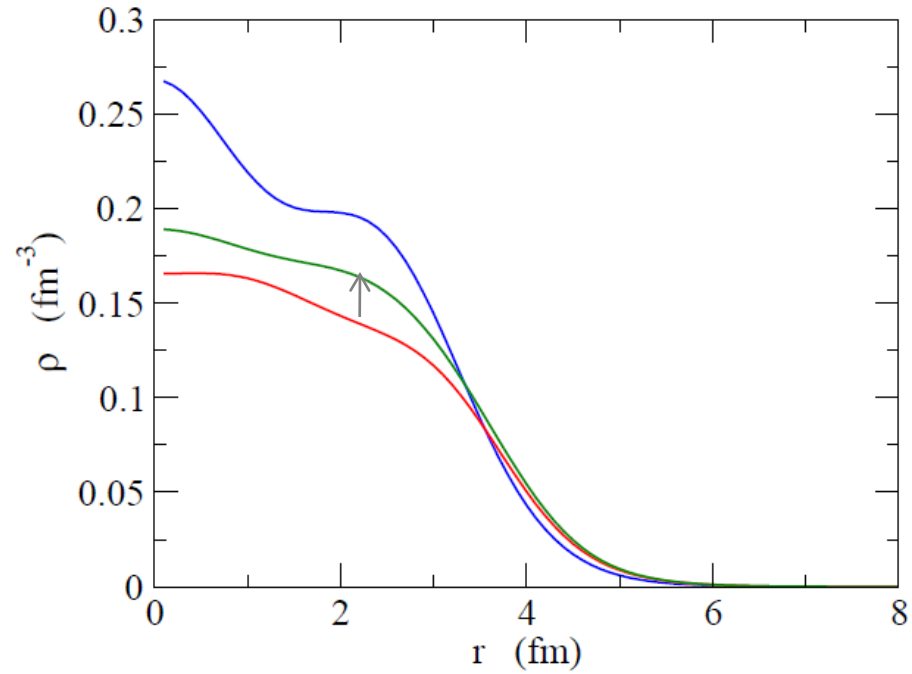
optimize the density by taking into account the nucleon-nucleon interaction

Skyrme-Hartree-Fock calculations for ^{40}Ca



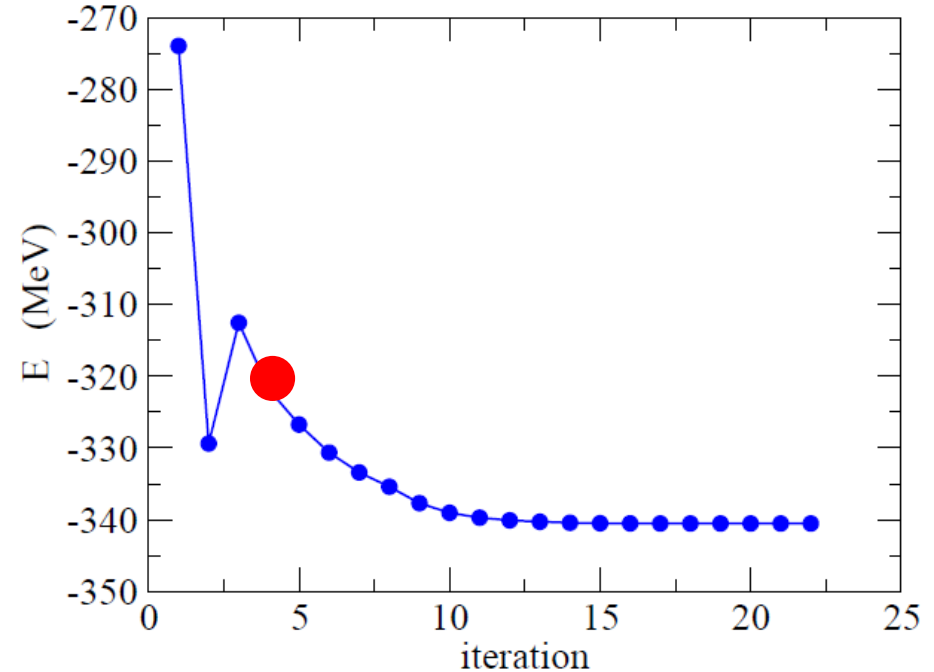
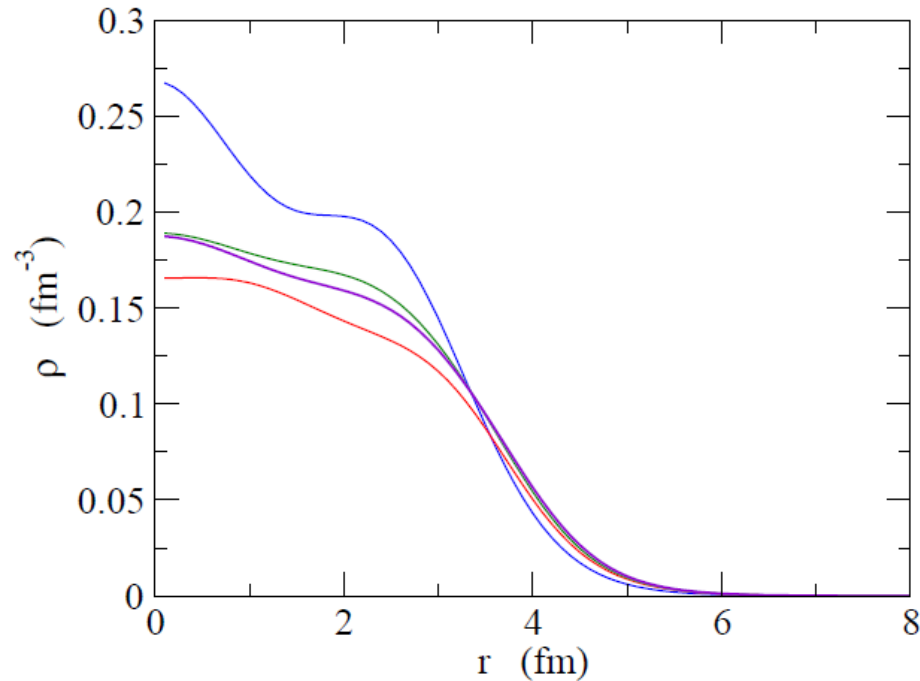
optimize the density by taking into account the nucleon-nucleon interaction

Skyrme-Hartree-Fock calculations for ^{40}Ca



optimize the density by taking into account the
nucleon-nucleon interaction

Skyrme-Hartree-Fock calculations for ^{40}Ca



optimize the density by taking into account the nucleon-nucleon interaction



optimized density (and shape) can be determined automatically

Variational Principle

(Rayleigh-Ritz method)

optimization \longleftrightarrow variational principle

$$\frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle} \geq E_{\text{g.s.}}$$

$$|\Psi\rangle = \sum_n C_n |\phi_n\rangle$$
$$\longrightarrow \text{lhs} = \frac{\sum_n C_n^2 E_n}{\sum_n C_n^2} \geq E_0$$

H : many-body Hamiltonian

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots) = \psi_1(\mathbf{r}_1) \cdot \psi_2(\mathbf{r}_2) \cdot \psi_3(\mathbf{r}_3) \cdot \dots$$

\longleftarrow many-body wave function for independent particles

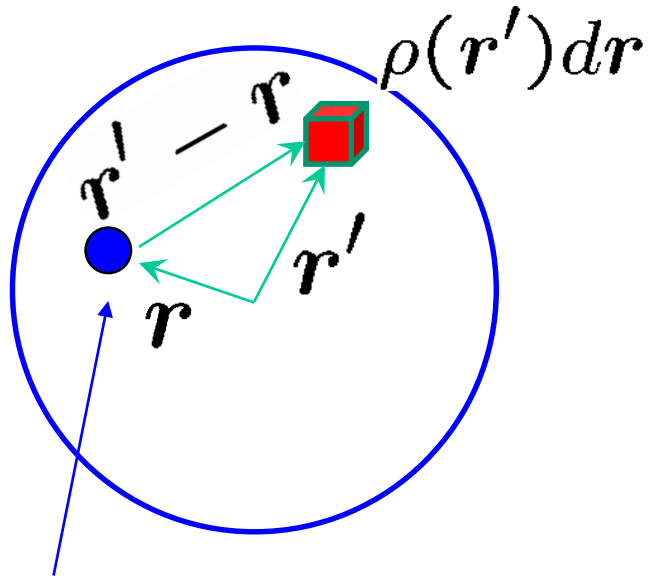


$$\left[-\frac{\hbar^2}{2m} \nabla^2 + \int v(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') d\mathbf{r}' - \epsilon_i \right] \psi_i(\mathbf{r}) = 0$$

change gradually the single-particle potential so that the total energy becomes minimum

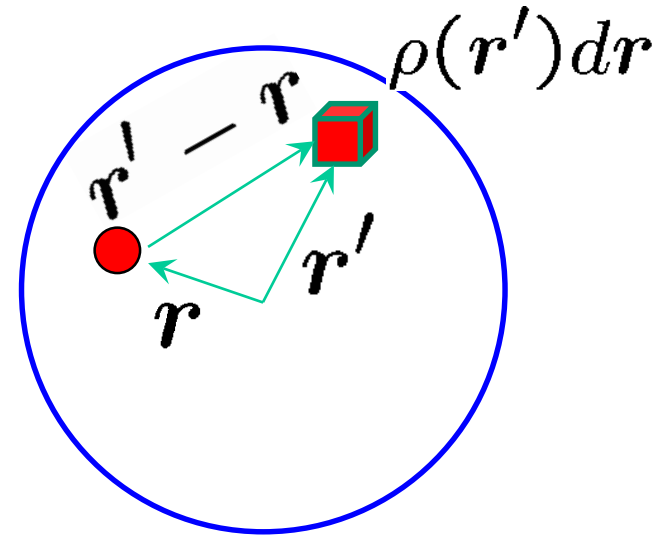
Mean-field (Hartree-Fock) Theory

electro-static potential



test charge

nucleus




interaction between identical particles
→ needs anti-symmetrization

$$V(\mathbf{r}) \sim \int v(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}'$$


anti-symmetrization

nucleon: fermion


$$\Psi(x_1, x_2, x_3 \dots) = -\Psi(x_2, x_1, x_3 \dots)$$

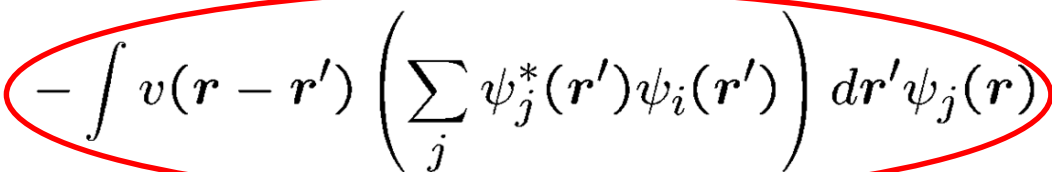
$$\psi_1(x_1)\psi_2(x_2) \rightarrow [\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2)]$$

Slater determinat


$$0 = \left[-\frac{\hbar^2}{2m} \nabla^2 + \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j |\psi_j(\mathbf{r}')|^2 \right) d\mathbf{r}' - \epsilon_i \right] \psi_i(\mathbf{r})$$

$$\psi_j^*(\mathbf{r}')\psi_j(\mathbf{r}')\psi_i(\mathbf{r}) \rightarrow \psi_j^*(\mathbf{r}')\psi_i(\mathbf{r}')\psi_j(\mathbf{r})$$

$$\rightarrow \left[-\frac{\hbar^2}{2m} \nabla^2 + \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j |\psi_j(\mathbf{r}')|^2 \right) d\mathbf{r}' - \epsilon_i \right] \psi_i(\mathbf{r})$$


$$- \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j \psi_j^*(\mathbf{r}')\psi_i(\mathbf{r}') \right) d\mathbf{r}' \psi_j(\mathbf{r})$$

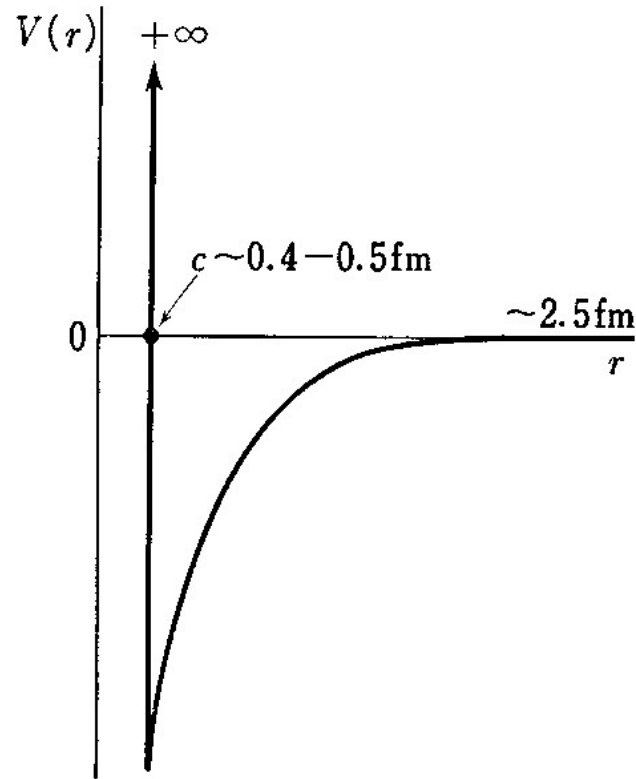
exchange term

anti-symmetrization

$$\begin{aligned} 0 &= \left[-\frac{\hbar^2}{2m} \nabla^2 + \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j |\psi_j(\mathbf{r}')|^2 \right) d\mathbf{r}' - \epsilon_i \right] \psi_i(\mathbf{r}) \\ &\rightarrow \left[-\frac{\hbar^2}{2m} \nabla^2 + \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j |\psi_j(\mathbf{r}')|^2 \right) d\mathbf{r}' - \epsilon_i \right] \psi_i(\mathbf{r}) \\ &\quad - \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j \psi_j^*(\mathbf{r}') \psi_i(\mathbf{r}') \right) d\mathbf{r}' \psi_j(\mathbf{r}) \\ &= \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) - \epsilon_i \right] \psi_i(\mathbf{r}) + \int d\mathbf{r}' V_{\text{NL}}(\mathbf{r}, \mathbf{r}') \psi_i(\mathbf{r}') \end{aligned}$$

non-local potential

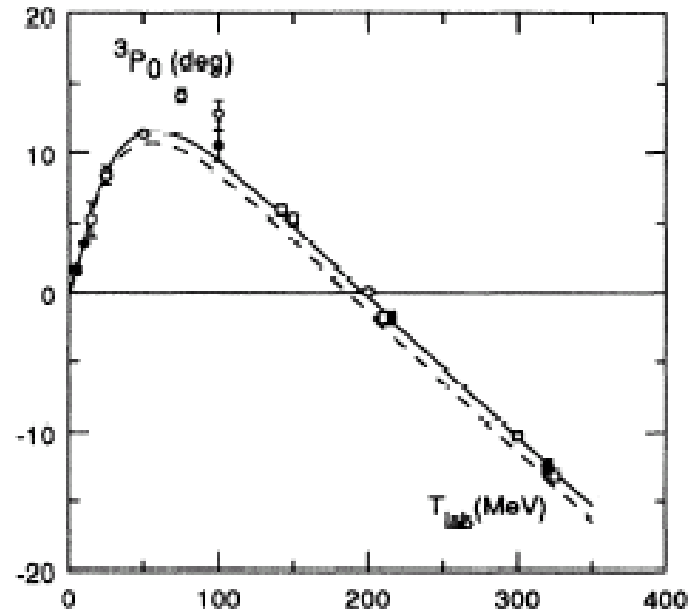
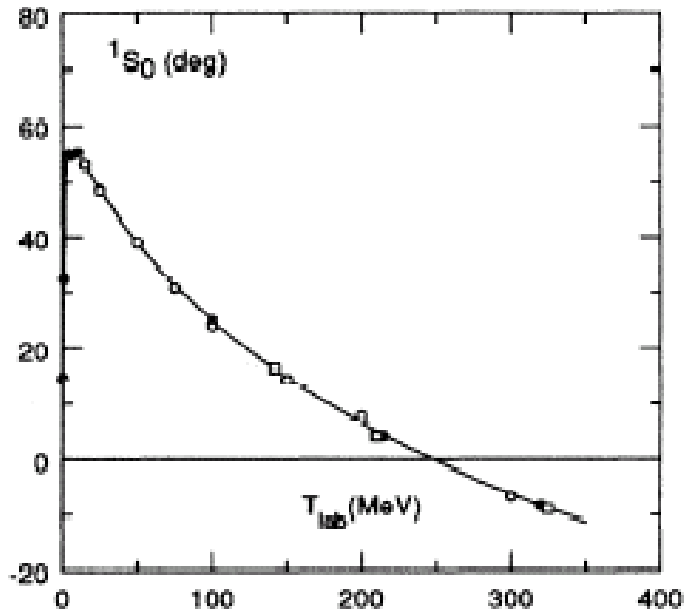
Bare nucleon-nucleon interaction



Existence of short range
repulsive core

Bare nucleon-nucleon interaction

Phase shift for p-p scattering



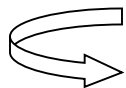
(V.G.J. Stoks et al., PRC48('93)792)

Phase shift: +ve \rightarrow -ve
at high energies

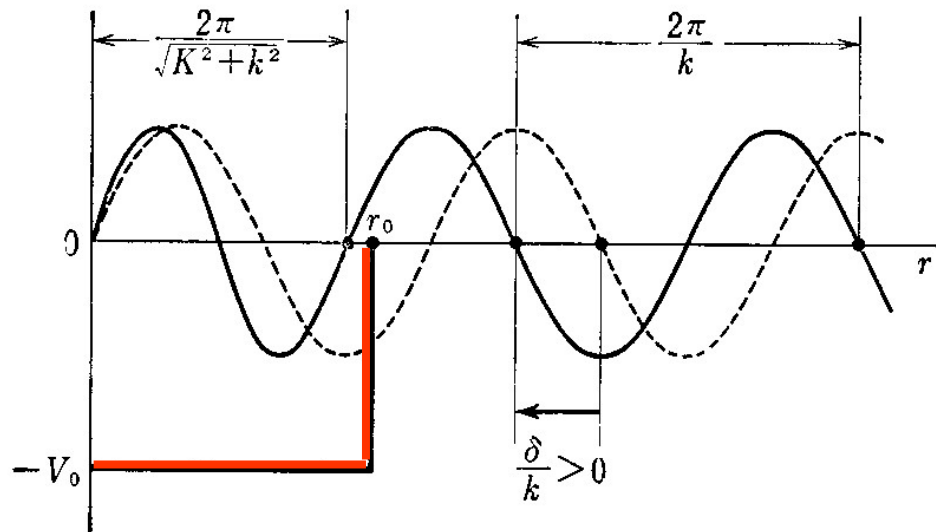
Phase shift:

Radial wave function

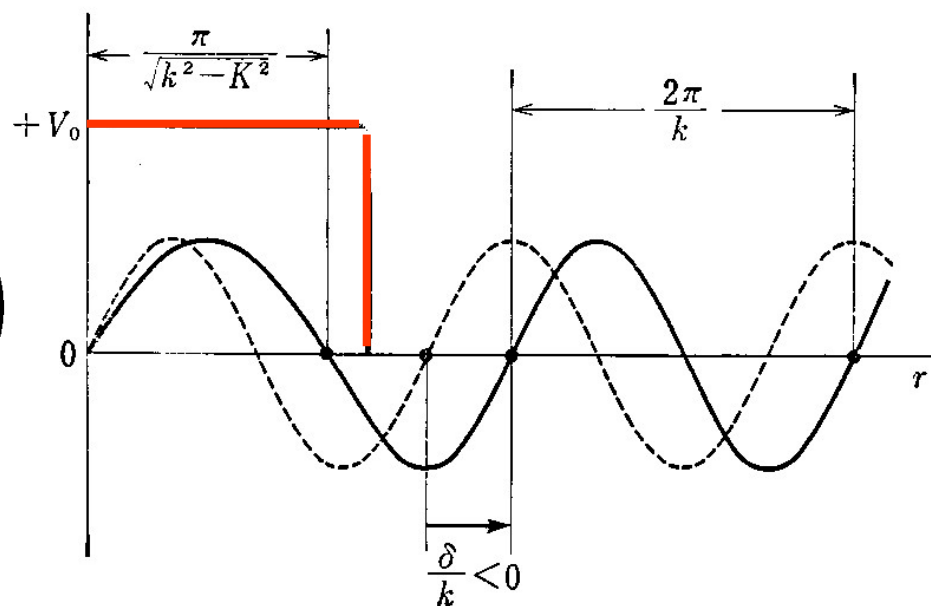
$$\Psi_l(r) = \frac{u_l(r)}{r} Y_{lm}(\hat{r})$$



$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V(r) + \frac{l(l+1)\hbar^2}{2mr^2} - E \right] u_l(r) = 0$$



(a) 引力 attraction

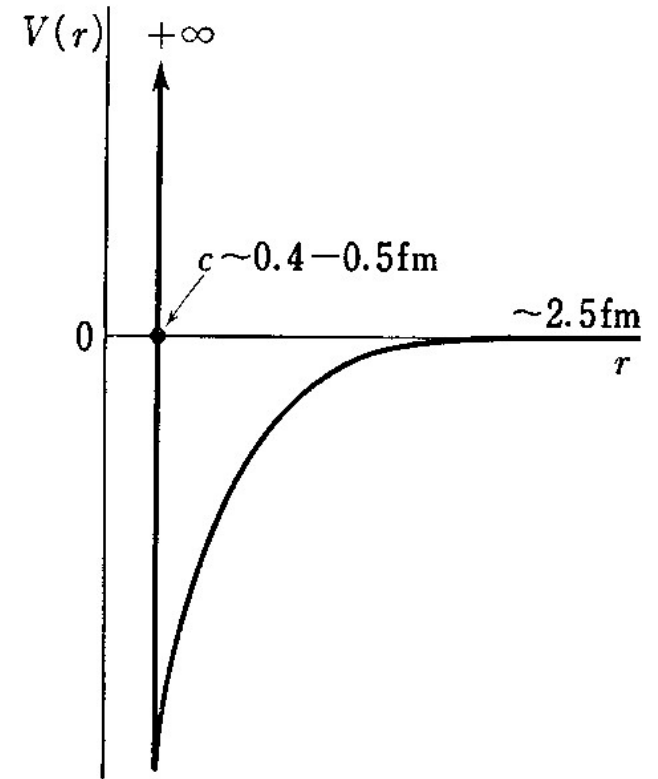
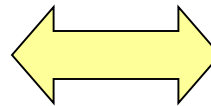
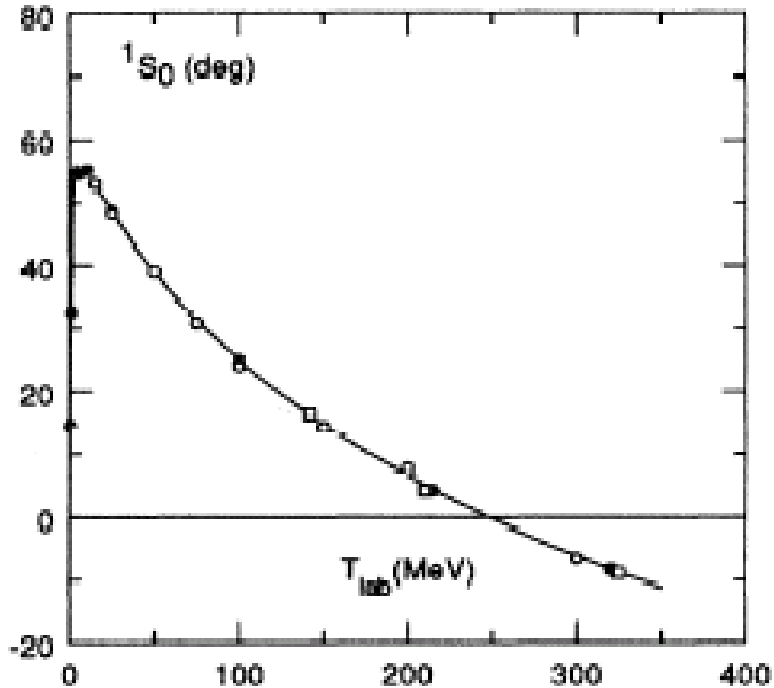


(b) 斥力 repulsion

Asymptotic form:

$$u_l(r) \rightarrow \sin\left(kr - \frac{l\pi}{2} + \delta_l\right) \quad (r \rightarrow \infty)$$

$\delta > 0 \rightarrow$ attraction
 $\delta < 0 \rightarrow$ repulsion



Phase shift: +ve \rightarrow -ve
 at high energies

Existence of short range
 repulsive core

Bruckner's G-matrix Nucleon-nucleon interaction *in medium*

Nucleon-nucleon interaction with a hard core

→ HF method: does not work

← Matrix elements: diverge

.....but the HF picture seems to work in nuclear systems

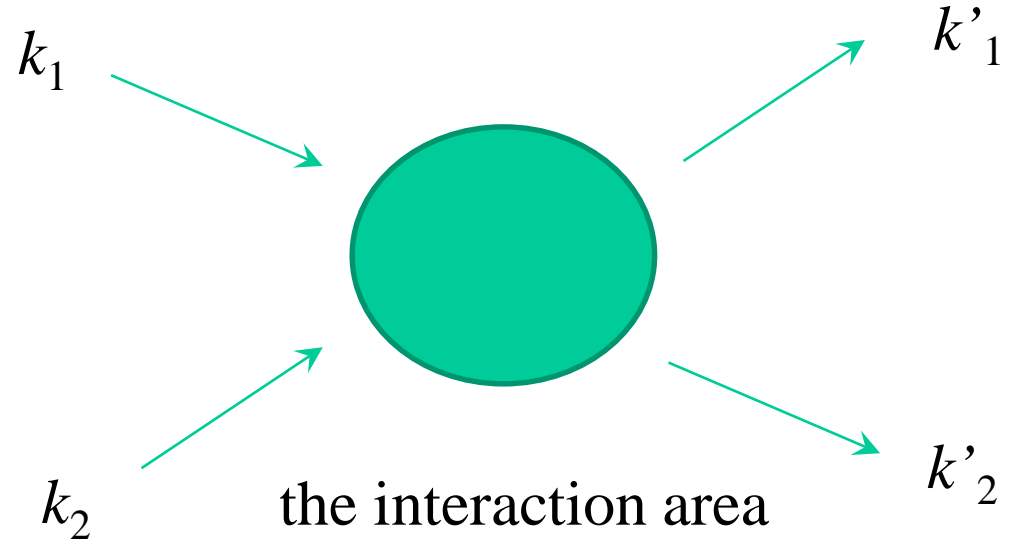
cf. magic numbers

Solution: a nucleon-nucleon interaction *in medium* (effective interaction) rather than a bare interaction



Bruckner's G-matrix

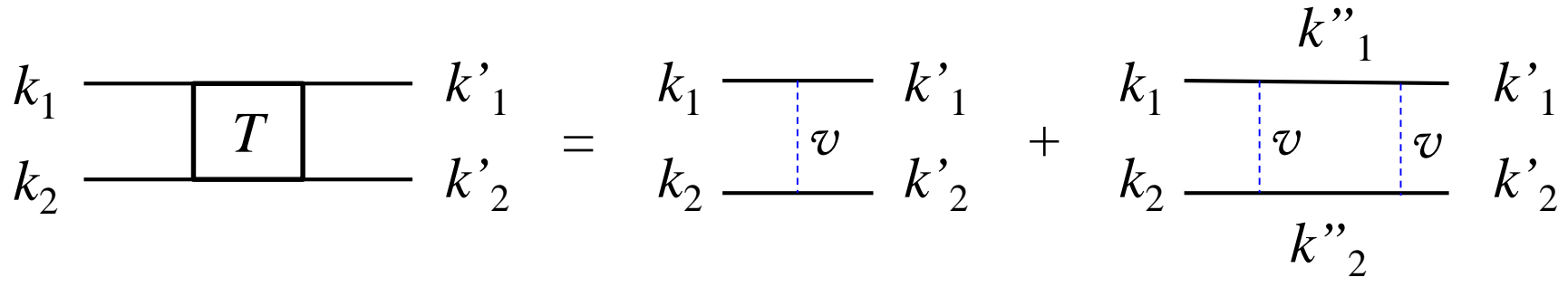
➤ two-body (multiple) scattering *in vacuum*



$$\begin{array}{c}
 k_1 \text{ --- } \boxed{T} \text{ --- } k'_1 \\
 k_2 \text{ --- } \boxed{T} \text{ --- } k'_2
 \end{array}
 =
 \begin{array}{c}
 k_1 \text{ --- } \text{---} \text{---} k'_1 \\
 | \quad \quad \quad | \\
 \nu \quad \quad \quad \nu \\
 k_2 \text{ --- } \text{---} \text{---} k'_2
 \end{array}
 +
 \begin{array}{c}
 k_1 \text{ --- } \text{---} k''_1 \text{ ---} \text{---} k'_1 \\
 | \quad \quad \quad | \quad \quad \quad | \\
 \nu \quad \quad \quad \nu \quad \quad \quad \nu \\
 k_2 \text{ --- } \text{---} k''_2 \text{ ---} \text{---} k'_2
 \end{array}
 + \dots$$

the 1st order
the 2nd order
higher orders

➤ two-body (multiple) scattering *in vacuum*



+.....

Lippmann-Schwinger equation

$$T = v + v \frac{1}{E - H_0} T$$

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V - E \right) \psi = 0$$

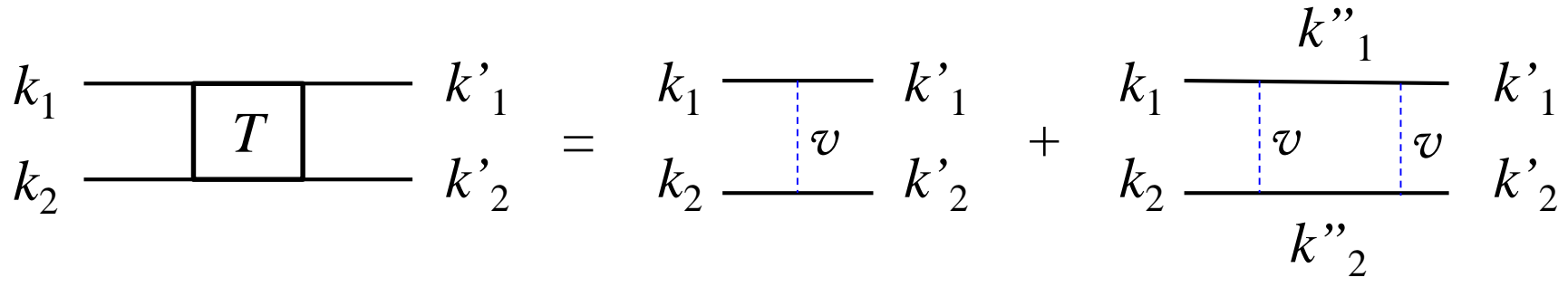
➡
$$\left(-\frac{\hbar^2}{2m} \nabla^2 - E \right) \psi = -V \psi$$

➡
$$\psi = \phi - \frac{1}{H_0 - E} V \psi \quad H_0 = -\frac{\hbar^2}{2m} \nabla^2, \quad (H_0 - E) \phi = 0$$

➡
$$V \psi = V \phi - V \frac{1}{H_0 - E} V \psi \quad \Rightarrow \quad T = V - V \frac{1}{H_0 - E} T$$

 ($V \psi = T \phi$)

➤ two-body (multiple) scattering *in vacuum*



+.....

Lippmann-Schwinger equation

$$T = v + v \frac{1}{E - H_0} T$$

➤ two-body (multiple) scattering *in vacuum*

$$\begin{array}{c}
 k_1 \text{ --- } \boxed{T} \text{ --- } k'_1 \\
 k_2 \text{ --- } \text{ --- } k'_2
 \end{array}
 =
 \begin{array}{c}
 k_1 \text{ --- } \text{---} v \text{ ---} k'_1 \\
 k_2 \text{ --- } \text{---} \text{ --- } k'_2
 \end{array}
 +
 \begin{array}{c}
 k_1 \text{ --- } \boxed{k''_1} \text{ ---} \\
 \text{---} v \text{ ---} v \text{ ---} \\
 k_2 \text{ --- } \text{---} \boxed{k''_2} \text{ ---} k'_2
 \end{array}
 + \dots$$

Lippmann-Schwinger equation

➤ two-body (multiple) scattering *in medium*

$$\begin{array}{c}
 k_1 \text{ --- } \boxed{G} \text{ --- } k'_1 \\
 k_2 \text{ --- } \text{ --- } k'_2
 \end{array}
 =
 \begin{array}{c}
 k_1 \text{ --- } \text{---} v \text{ ---} k'_1 \\
 k_2 \text{ --- } \text{---} \text{ --- } k'_2
 \end{array}
 +
 \begin{array}{c}
 k_1 \text{ --- } \boxed{k''_1 > k_F} \text{ ---} \\
 \text{---} v \text{ ---} v \text{ ---} \\
 k_2 \text{ --- } \text{---} \boxed{k''_2 > k_F} \text{ ---} k'_2
 \end{array}
 + \dots$$

Pauli principle

*scattering: suppressed
 because intermediate states have to have
 $k > k_F \rightarrow$ independent particle picture

Bethe-Goldstone equation

$$G = v + v \frac{Q_F}{E - H_0} G$$

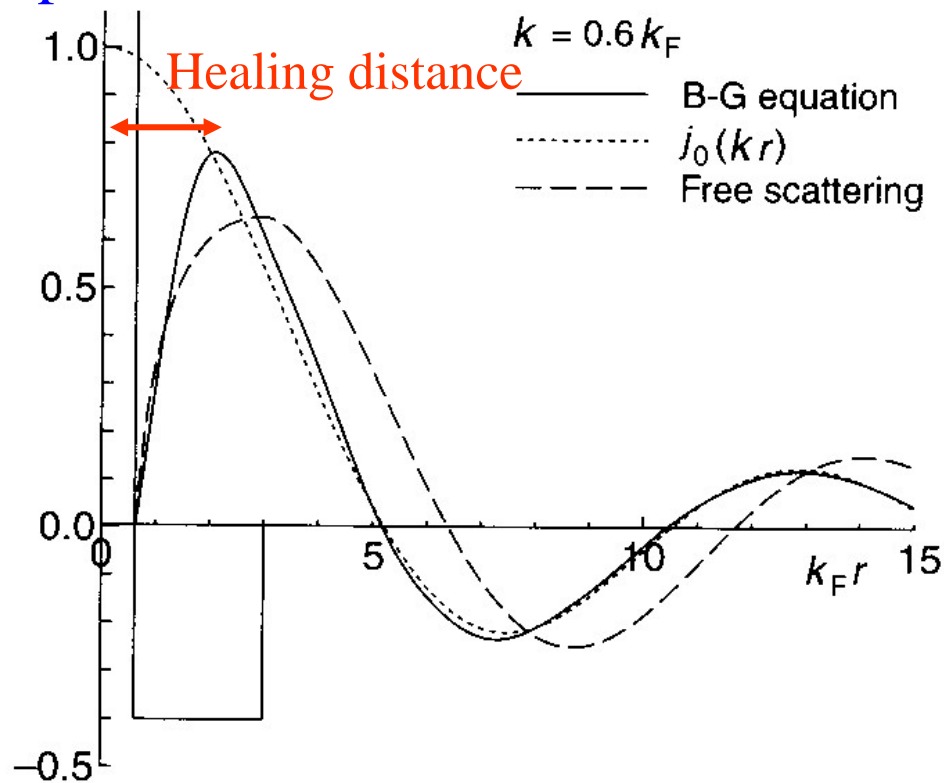
◆ Hard core

$$G = v + v \frac{Q_F}{E - H_0} G \iff G = \frac{v}{1 - v \frac{Q_F}{E - H_0}}$$

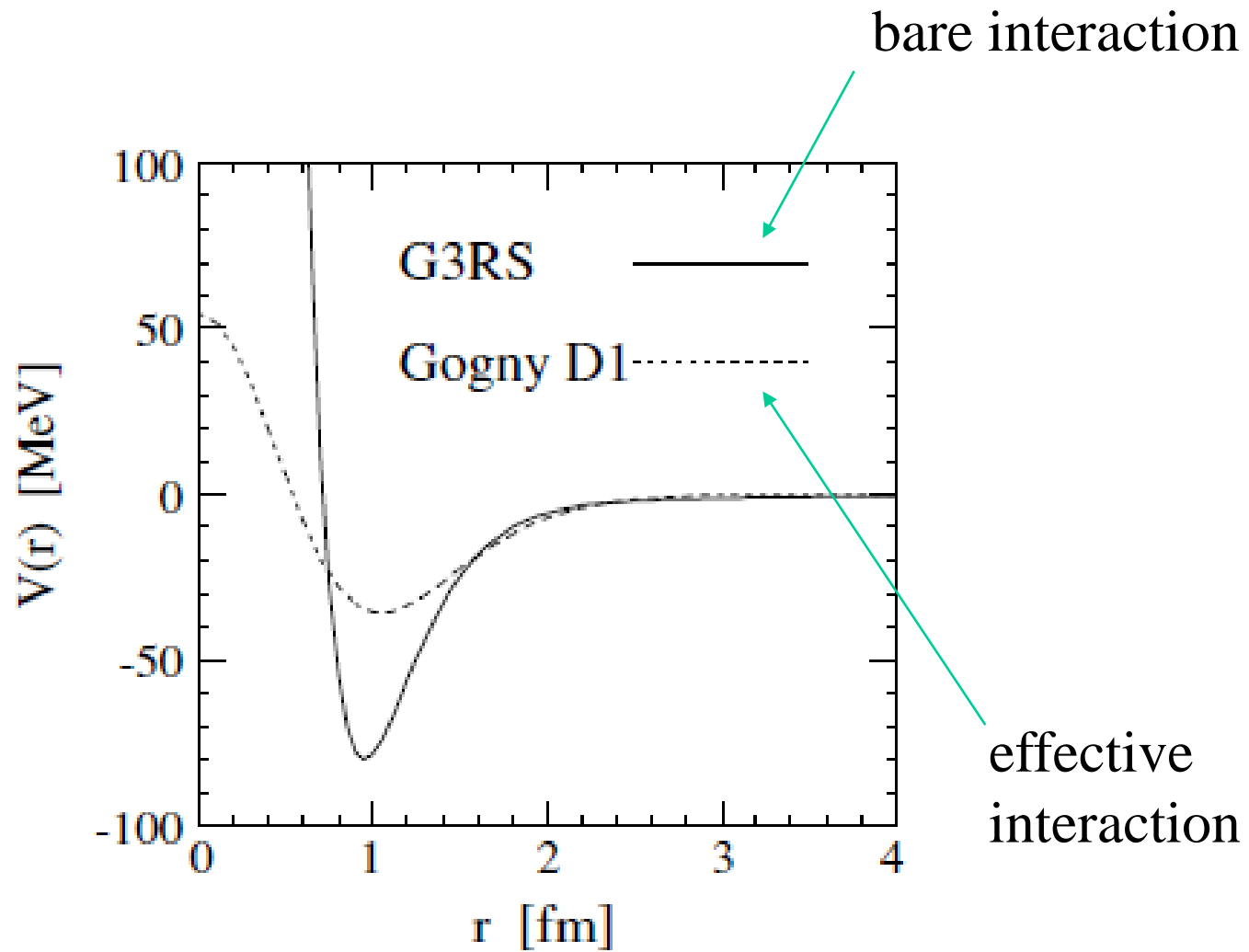


Even if v tends to infinity, G may stay finite.

◆ Independent particle motion



→ use G instead of v in mean-field calculations



M. Matsuo, Phys. Rev. C73('06)044309

Phenomenological effective interactions

G-matrix

- ab initio
- but, cumbersome to compute (especially for finite nuclei)
- qualitatively good, but quantitatively not successful



HF calculations with a phenomenological effective interaction

Philosophy: take the functional form of G , but determine the parameters phenomenologically

- Skyrme interaction (non-rel., zero range)
- Gogny interaction (non-rel., finite range)
- Relativistic mean-field model (relativistic, “meson exchanges”)

Skyrme interaction density dependent zero-range interaction

$$\begin{aligned}
 v(\mathbf{r}, \mathbf{r}') &= t_0(1 + x_0\hat{P}_\sigma)\delta(\mathbf{r} - \mathbf{r}') \\
 &+ \frac{1}{2}t_1(1 + x_1\hat{P}_\sigma)(\mathbf{k}^2\delta(\mathbf{r} - \mathbf{r}') + \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}^2) \\
 &+ t_2(1 + x_2\hat{P}_\sigma)\mathbf{k}\delta(\mathbf{r} - \mathbf{r}')\mathbf{k} \\
 &+ \frac{1}{6}t_3(1 + x_3\hat{P}_\sigma)\delta(\mathbf{r} - \mathbf{r}')\rho^\alpha((\mathbf{r} + \mathbf{r}')/2) \\
 &+ iW_0(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \mathbf{k} \times \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}
 \end{aligned}$$

if $x_i=0, t_1=t_2=0$:

$$\mathbf{k} = (\nabla_1 - \nabla_2)/2i$$

$$v(\mathbf{r}, \mathbf{r}') = \underbrace{t_0\delta(\mathbf{r} - \mathbf{r}')}_{\text{short-range attraction}} + \underbrace{\frac{1}{6}t_3\delta(\mathbf{r} - \mathbf{r}')\rho^\alpha(\mathbf{r})}_{\text{repulsion to avoid collapse}}$$

$$\underbrace{+iW_0(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \mathbf{k} \times \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}}_{\text{spin-orbit interaction}}$$

spin-orbit interaction

Skyrme interaction density dependent zero-range interaction

$$\begin{aligned}v(\mathbf{r}, \mathbf{r}') &= t_0(1 + x_0\hat{P}_\sigma)\delta(\mathbf{r} - \mathbf{r}') \\ &+ \frac{1}{2}t_1(1 + x_1\hat{P}_\sigma)(\mathbf{k}^2\delta(\mathbf{r} - \mathbf{r}') + \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}^2) \\ &+ t_2(1 + x_2\hat{P}_\sigma)\mathbf{k}\delta(\mathbf{r} - \mathbf{r}')\mathbf{k} \\ &+ \frac{1}{6}t_3(1 + x_3\hat{P}_\sigma)\delta(\mathbf{r} - \mathbf{r}')\rho^\alpha((\mathbf{r} + \mathbf{r}')/2) \\ &+ iW_0(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \mathbf{k} \times \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}\end{aligned}$$

$$\mathbf{k} = (\nabla_1 - \nabla_2)/2i$$

(note) finite range effect \longleftrightarrow momentum dependence

$$\begin{aligned}\langle \mathbf{p} | V | \mathbf{p}' \rangle &= \frac{1}{(2\pi\hbar)^3} \int d\mathbf{r} e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}/\hbar} V(\mathbf{r}) \\ &\sim V_0 + V_1(\mathbf{p}^2 + \mathbf{p}'^2) + V_2\mathbf{p}\mathbf{p}' + \dots \\ &\rightarrow V_0\delta(\mathbf{r}) + V_1(\hat{\mathbf{p}}^2\delta(\mathbf{r}) + \delta(\mathbf{r})\hat{\mathbf{p}}^2) + V_2\hat{\mathbf{p}}\delta(\mathbf{r})\hat{\mathbf{p}}\end{aligned}$$

Skyrme interaction density dependent zero-range interaction

$$\begin{aligned}v(\mathbf{r}, \mathbf{r}') &= t_0(1 + x_0\hat{P}_\sigma)\delta(\mathbf{r} - \mathbf{r}') \\ &+ \frac{1}{2}t_1(1 + x_1\hat{P}_\sigma)(\mathbf{k}^2\delta(\mathbf{r} - \mathbf{r}') + \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}^2) \\ &+ t_2(1 + x_2\hat{P}_\sigma)\mathbf{k}\delta(\mathbf{r} - \mathbf{r}')\mathbf{k} \\ &+ \frac{1}{6}t_3(1 + x_3\hat{P}_\sigma)\delta(\mathbf{r} - \mathbf{r}')\rho^\alpha((\mathbf{r} + \mathbf{r}')/2) \\ &+ iW_0(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \mathbf{k} \times \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}\end{aligned}$$

$$\mathbf{k} = (\nabla_1 - \nabla_2)/2i$$

the exchange potential \longrightarrow local

$$\begin{aligned}0 &= \left[-\frac{\hbar^2}{2m}\nabla^2 + \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j |\psi_j(\mathbf{r}')|^2 \right) d\mathbf{r}' - \epsilon_i \right] \psi_i(\mathbf{r}) \\ &- \int v(\mathbf{r} - \mathbf{r}') \left(\sum_j \psi_j^*(\mathbf{r}')\psi_i(\mathbf{r}') \right) d\mathbf{r}'\psi_j(\mathbf{r})\end{aligned}$$

Skyrme interactions: 10 adjustable parameters

$$\begin{aligned}v(\mathbf{r}, \mathbf{r}') &= t_0(1 + x_0\hat{P}_\sigma)\delta(\mathbf{r} - \mathbf{r}') \\ &+ \frac{1}{2}t_1(1 + x_1\hat{P}_\sigma)(\mathbf{k}^2\delta(\mathbf{r} - \mathbf{r}') + \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}^2) \\ &+ t_2(1 + x_2\hat{P}_\sigma)\mathbf{k}\delta(\mathbf{r} - \mathbf{r}')\mathbf{k} \\ &+ \frac{1}{6}t_3(1 + x_3\hat{P}_\sigma)\delta(\mathbf{r} - \mathbf{r}')\rho^\alpha((\mathbf{r} + \mathbf{r}')/2) \\ &+ iW_0(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \mathbf{k} \times \delta(\mathbf{r} - \mathbf{r}')\mathbf{k}\end{aligned}$$

A fitting strategy:

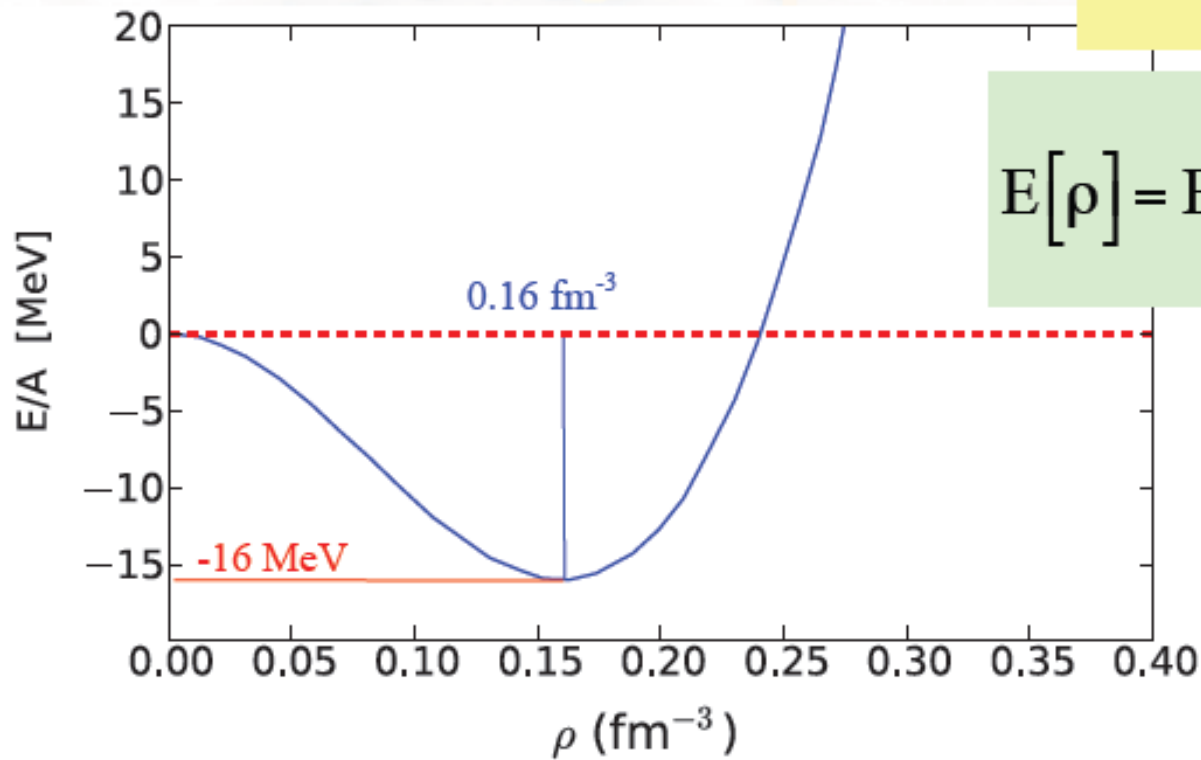
B.E. and r_{rms} : ^{16}O , ^{40}Ca , ^{48}Ca , ^{56}Ni , ^{90}Zr , ^{208}Pb ,.....

Infinite nuclear matter: E/A , ρ_{eq} ,.....

Parameter sets:

SIII, SkM*, SGII, SLy4,.....

EOS of infinite nuclear matter



$$K_{\infty} = 9\rho^2 \left. \frac{d^2[E(\rho)/\rho]}{d\rho^2} \right|_{\rho_0}$$

$$E[\rho] = E[\rho_0] + \frac{1}{18} K_{\infty} \left(\frac{\rho - \rho_0}{\rho_0} \right)^2$$

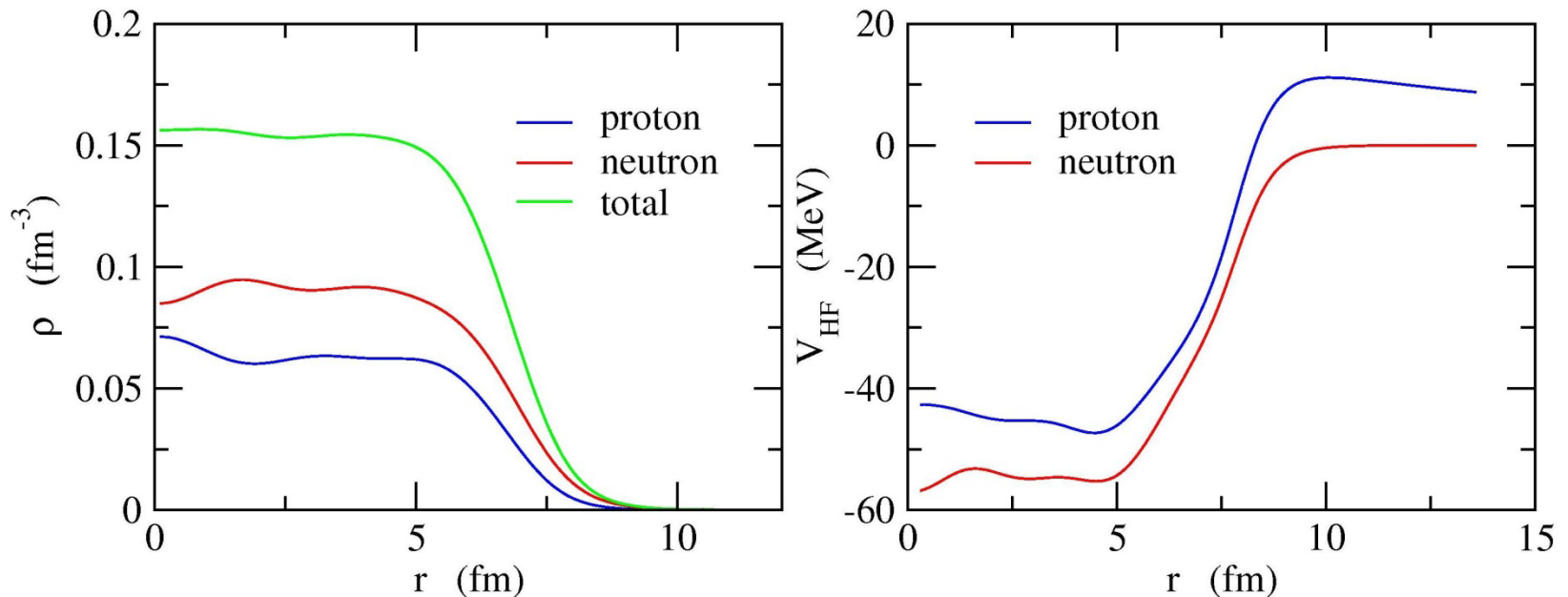
$$-\frac{\hbar^2}{2m} \nabla^2 \psi_i(\mathbf{r}) + \int v(\mathbf{r}, \mathbf{r}') \rho_{\text{HF}}(\mathbf{r}') d\mathbf{r}' \psi_i(\mathbf{r}) - \int \rho_{\text{HF}}(\mathbf{r}, \mathbf{r}') v(\mathbf{r}, \mathbf{r}') \psi_i(\mathbf{r}') d\mathbf{r}' = \epsilon_i \psi_i(\mathbf{r})$$

Iteration

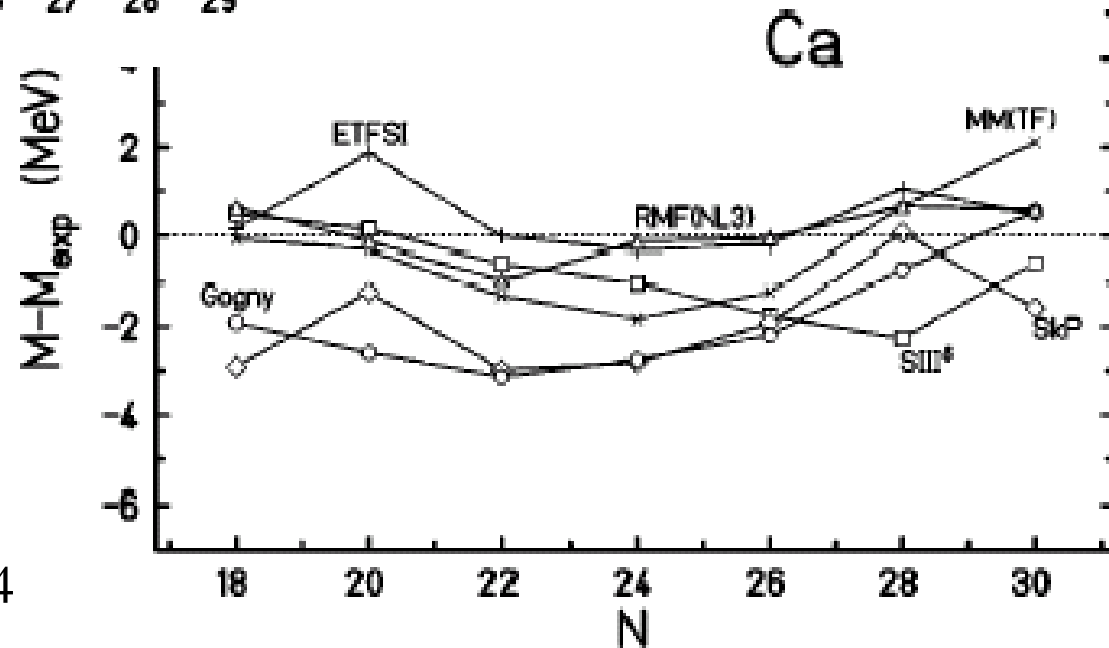
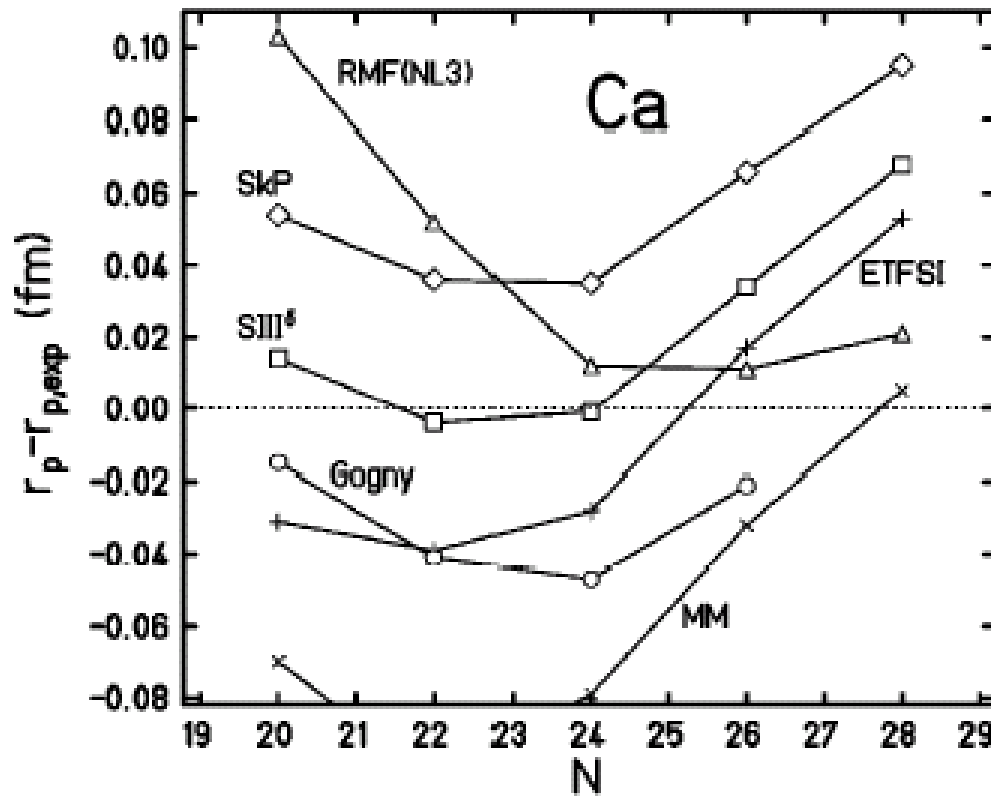
V_{HF} : depends on ψ_i ← non-linear problem

Iteration: $\{\psi_i\} \rightarrow \rho_{\text{HF}} \rightarrow V_{\text{HF}} \rightarrow \{\psi_i\} \rightarrow \dots$

^{208}Pb (Skyrme Hartree-Fock with SKM*)



Examples of HF calculations
for masses and radii



Z. Patyk et al.,
PRC59('99)704

deformation and two-neutron separation energy

