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1. 3次元球対称ポテンシャル中の運動

1次元ポテンシャル中の運動 ← 量子力学 I

$$\underbrace{\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right)}_{\text{ハミルトニアン } H = \frac{p^2}{2m}+V(x)}\psi(x) = E\,\psi(x)$$

$$\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = 1 \quad (\text{規格化条件})$$

⤳ 3次元ポテンシャルへの拡張

$$H = \frac{1}{2m}\left(p_x^2+p_y^2+p_z^2\right)+V(x,y,z)$$
$$= \frac{\mathbf{p}^2}{2m}+V(\mathbf{r})$$

座標表示： $\mathbf{p} = \frac{\hbar}{i}\nabla \qquad \left(p_k = \frac{\hbar}{i}\frac{\partial}{\partial x_k}\right)$

$$H = -\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r})$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

シュレーディンガー方程式：

$$\left(-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r})\right)\psi(\mathbf{r}) = E\,\psi(\mathbf{r})$$

波動関数の規格化：

$$\underbrace{\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz}_{=\int d\mathbf{r}}\,|\psi(\mathbf{r})|^2 = 1$$

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ポテンシャルが変数分離型のとき，

例）３次元調和振動子

$$V(x,y,z) = \frac{1}{2}m\omega_x^2 x^2 + \frac{1}{2}m\omega_y^2 y^2 + \frac{1}{2}m\omega_z^2 z^2$$

↓

$$H = \underbrace{\frac{p_x^2}{2m} + \frac{1}{2}m\omega_x x^2}_{= h_x} + \underbrace{\frac{p_y^2}{2m} + \frac{1}{2}m\omega_y y^2}_{= h_y} + \underbrace{\frac{p_z^2}{2m} + \frac{1}{2}m\omega_z z^2}_{= h_z}$$

シュレーディンガー方程式の解

$$\Psi(x,y,z) = \phi_{n_x}^{(1)}(x)\,\phi_{n_y}^{(2)}(y)\,\phi_{n_z}^{(3)}(z)$$

$$\left(\frac{p_x^2}{2m} + \frac{1}{2}m\omega_x x^2\right)\phi_{n_x}^{(1)}(x) = \left(n_x + \frac{1}{2}\right)\hbar\omega_x \times \phi_{n_x}^{(1)}(x)$$

など．

$$E = \left(n_x + \frac{1}{2}\right)\hbar\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\omega_y + \left(n_z + \frac{1}{2}\right)\hbar\omega_z$$

(note) $\omega_x = \omega_y = \omega_z \equiv \omega$ のとき

$$E = \left(n_x + n_y + n_z + \frac{3}{2}\right)\hbar\omega$$

$$V(x,y,z) = \frac{1}{2}m\omega^2 \underbrace{(x^2 + y^2 + z^2)}_{= r^2}$$

→ 極座標を使っても解ける．

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(複習) 変数分離

$$H = \underbrace{\frac{P_x^2}{2m} + V(x)}_{= h_x} + \underbrace{\frac{P_y^2}{2m} + V(y)}_{= h_y}$$

$$H \Psi(x, y) = E \Psi(x, y)$$

$\Psi(x, y) = X(x) Y(y)$ とおく。

$$H \Psi = (h_x X) Y + X (h_y Y)$$

$$h_x X = E_x X$$
$$h_y Y = E_y Y$$

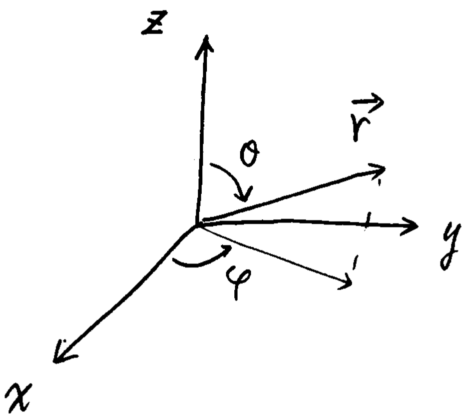
のとき

$$H \Psi = (E_x + E_y) X \cdot Y = \underbrace{(E_x + E_y)}_{= E} \Psi$$

ポテンシャルが球対称の時（$\mathbf{r}$の大きさ $r=|\mathbf{r}|$ にしか依存しない場合）

$$H = \frac{\mathbb{P}^2}{2m} + V(r)$$

$\longrightarrow$ 極座標表示をすることにより
変数分離が可能



$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$
$$\left(r = \sqrt{x^2+y^2+z^2} \right)$$

微分オペレーター ∇ を極座標を使って書く：

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}$$

(note) $r = \sqrt{x^2+y^2+z^2} \longrightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

(note) $z = r\cos\theta$ の両辺を x で微分すると

$$0 = \underbrace{\frac{\partial r}{\partial x}}_{=\frac{x}{r}}\cos\theta - r\sin\theta\frac{\partial \theta}{\partial x}$$

$$\rightarrow \frac{\partial \theta}{\partial x} = \frac{x\cos\theta}{r^2\sin\theta} = \frac{1}{r}\cos\theta\cos\varphi$$

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(note) $y = r\sin\theta\sin\varphi$ の両辺を x で微分

$\downarrow$

$$0 = \frac{\partial r}{\partial x}\sin\theta\sin\varphi + r\cos\theta\frac{\partial\theta}{\partial x}\sin\varphi + r\sin\theta\cos\varphi\frac{\partial\varphi}{\partial x}$$

$$= \sin^2\theta\,\underline{\sin\varphi\cos\varphi} + \cos^2\theta\,\underline{\cos\varphi\sin\varphi} + \underline{r\sin\theta}\cos\varphi\frac{\partial\varphi}{\partial x}$$

$$= \sin\varphi\cancel{\cos\varphi} + r\sin\theta\cancel{\cos\varphi}\frac{\partial\varphi}{\partial x}$$

$$\therefore \quad \frac{\partial\varphi}{\partial x} = -\frac{\sin\varphi}{r\sin\theta}$$

$$\therefore \quad \frac{\partial}{\partial x} = \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

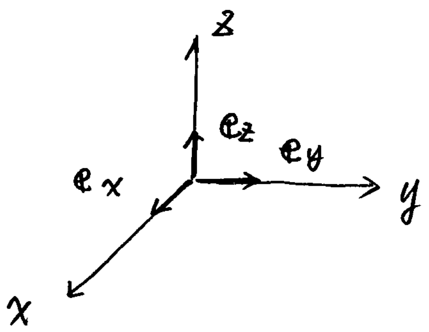
$\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ も同様に求められる。

$\downarrow$

$$\nabla = \frac{\partial}{\partial x}\mathbf{e}_x + \frac{\partial}{\partial y}\mathbf{e}_y + \frac{\partial}{\partial z}\mathbf{e}_z$$

$$= \left[\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right]\mathbf{e}_x$$
$$+ \left[\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right]\mathbf{e}_y$$
$$+ \left[\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right]\mathbf{e}_z$$

$\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ は x, y, z 方向の単位ベクトル



$$|\mathbf{e}_k| = 1$$

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さらにもう一度微分演算を行なえば
∇^2の極座標表示が得られる（うしろのページ参照）

$$\boxed{\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\hat{\ell}^2$$

$$-\hat{\ell}^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

シュレーディンガー方程式：

$$\boxed{\left[-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\hat{\ell}^2\hbar^2}{2mr^2} + V(r) - E\right]\psi(\mathbf{r}) = 0}$$

角度に関係する演算子は$\hat{\ell}^2$のみ。

もし$\hat{\ell}^2$の固有状態　$\hat{\ell}^2 Y(\theta,\varphi) = \lambda Y(\theta,\varphi)$
がわかったとして

$$\psi(\mathbf{r}) = R(r)Y(\theta,\varphi)$$

と変数分離すれば

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\lambda\hbar^2}{2mr^2} + V(r) - E\right]R(r) = 0$$

となり，rのみの2階の微分方程式となる。

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[参考] ∇^2 の極座標表示の導出

$$\frac{\partial^2}{\partial x^2} = \left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)^2$$

$$= \sin^2\theta\cos^2\varphi\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\cos^2\theta\cos^2\varphi\frac{\partial^2}{\partial\theta^2} + \frac{\sin^2\varphi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$+ \frac{2}{r}\sin\theta\cos\theta\cos^2\varphi\frac{\partial^2}{\partial r\partial\theta} - \cancel{\frac{2}{r^2}\frac{\cos\theta}{\sin\theta}\sin\varphi\cos\varphi\frac{\partial^2}{\partial\theta\partial\varphi}}$$

$$- \cancel{\frac{2}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial r\partial\varphi}}$$

$$- \frac{1}{r^2}\sin\theta\cos\theta\cos^2\varphi\frac{\partial}{\partial\theta} + \cancel{\frac{1}{r^2}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}}$$

$$+ \frac{1}{r}\cos^2\theta\cos^2\varphi\frac{\partial}{\partial r} - \frac{1}{r^2}\sin\theta\cos\theta\cos^2\varphi\frac{\partial}{\partial\theta} + \cancel{\frac{\cos\theta\cos\varphi}{r}\cdot\frac{\sin\varphi}{r} \times \frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi}}$$

$$+ \frac{\sin\varphi}{r\cancel{\sin\theta}}\cancel{\sin\theta}\sin\varphi\frac{\partial}{\partial r} + \frac{\sin\varphi}{r\sin\theta}\cdot\frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta}$$

$$+ \cancel{\frac{\sin\varphi}{r\sin\theta}\cdot\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}}$$

$$\frac{\partial^2}{\partial y^2} = \left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)^2$$

$$= \sin^2\theta\sin^2\varphi\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\cos^2\theta\sin^2\varphi\frac{\partial^2}{\partial\theta^2} + \frac{\cos^2\varphi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$+ \frac{2}{r}\sin\theta\cos\theta\sin^2\varphi\frac{\partial^2}{\partial r\partial\theta} + \cancel{\frac{2}{r^2}\frac{\cos\theta}{\sin\theta}\sin\varphi\cos\varphi\frac{\partial^2}{\partial\theta\partial\varphi}}$$

$$+ \cancel{\frac{2}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial r\partial\varphi}}$$

$$- \frac{1}{r^2}\sin\theta\cos\theta\sin^2\varphi\frac{\partial}{\partial\theta} - \cancel{\frac{1}{r^2}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}}$$

$$+ \frac{1}{r}\cos^2\theta\sin^2\varphi\frac{\partial}{\partial r} - \frac{1}{r^2}\sin\theta\cos\theta\sin^2\varphi\frac{\partial}{\partial\theta} + \cancel{\frac{\cos\theta\sin\varphi}{r}\cdot\frac{\cos\varphi}{r}\left(-\frac{\cos\theta}{\sin^2\theta}\right)\frac{\partial}{\partial\varphi}}$$

$$+ \frac{\cos\varphi}{r\cancel{\sin\theta}}\cdot\cancel{\sin\theta}\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\cdot\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\theta}$$

$$- \cancel{\frac{\cos\varphi\sin\varphi}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}}$$

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$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = \sin^2\theta\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\cos^2\theta\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$
$$+\cancel{\frac{2}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial r\partial\theta}}-\cancel{\frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}}$$
$$+\frac{1}{r}\cos^2\theta\frac{\partial}{\partial r}-\cancel{\frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}}+\frac{1}{r}\frac{\partial}{\partial r}$$
$$+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}$$

$$\frac{\partial^2}{\partial z^2} = \left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)^2$$
$$=\cos^2\theta\frac{\partial^2}{\partial r^2}+\frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}-\cancel{\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}}$$
$$+\cancel{\frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}}+\frac{\sin^2\theta}{r}\frac{\partial}{\partial r}+\cancel{\frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}}$$

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+\frac{2}{r}\frac{\partial}{\partial r}$$
$$+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}$$

$$=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

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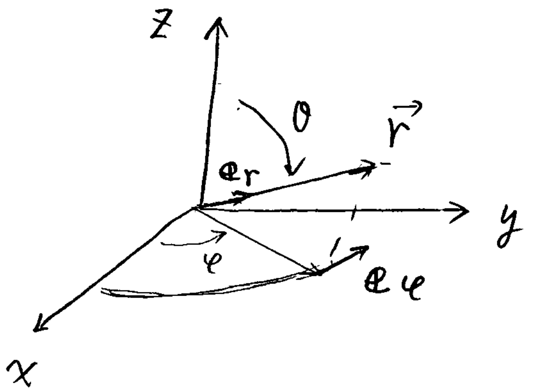
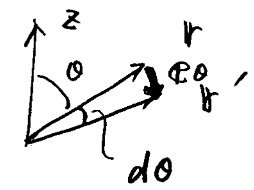
(note)

・e_x, e_y, e_z をさらに極座標を用いて表すと:



e_r : r の向きの単位ベクトル

e_θ :



$$e_r = \sin\theta\cos\varphi\, e_x + \sin\theta\sin\varphi\, e_y + \cos\theta\, e_z$$

$$\frac{\partial}{\partial\theta} e_r \rightarrow \quad e_\theta = \cos\theta\cos\varphi\, e_x + \cos\theta\sin\varphi\, e_y - \sin\theta\, e_z$$

$$e_\varphi = -\sin\varphi\, e_x + \cos\varphi\, e_y$$

(note) $$e_r \cdot e_\theta = \sin\theta\cos\theta\cos^2\varphi + \sin\theta\cos\theta\sin^2\varphi - \sin\theta\cos\theta = 0$$

同様に $e_r \cdot e_\varphi = e_\theta \cdot e_\varphi = 0$

$$\begin{aligned}\nabla \cdot e_r = {} & \sin^2\theta\cos^2\varphi\frac{\partial}{\partial r} + \cancel{\frac{1}{r}\sin\theta\cos\theta\cos^2\varphi\frac{\partial}{\partial\theta}} - \cancel{\frac{1}{r}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}} \\ & + \sin^2\theta\sin^2\varphi\frac{\partial}{\partial r} + \cancel{\frac{1}{r}\sin\theta\cos\theta\sin^2\varphi\frac{\partial}{\partial\theta}} + \cancel{\frac{1}{r}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}} \\ & + \cos^2\theta\frac{\partial}{\partial r} - \cancel{\frac{1}{r}\sin\theta\cos\theta\frac{\partial}{\partial\theta}} \\ = {} & \frac{\partial}{\partial r}\end{aligned}$$

$$\nabla \cdot e_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}$$

$$\nabla \cdot e_\varphi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\boxed{\nabla = e_r\frac{\partial}{\partial r} + \frac{1}{r}e_\theta\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}e_\varphi\frac{\partial}{\partial\varphi}}$$

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2. 角運動量

2.1. 角運動量演算子の極座標表示

$$\hat{\ell}^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

は角運動量の2乗を$\hbar^2$で割ったもの：

$$\mathbb{L}^2 = (\mathbb{r}\times\mathbb{p})^2 = \ell^2\hbar^2$$

(note)
$$L_z = x p_y - y p_x = \frac{\hbar}{i}\left(x\,\partial_y - y\,\partial_x\right)$$

$$= \frac{\hbar}{i}\Big\{ r\sin\theta\cos\varphi\left(\cancel{\sin\theta\sin\varphi}\,\partial_r + \frac{1}{r}\cos\theta\cancel{\sin\varphi}\,\partial_\theta + \frac{\cos\varphi}{r\sin\theta}\,\partial_\varphi\right)$$
$$- r\sin\theta\sin\varphi\left(\cancel{\sin\theta\cos\varphi}\,\partial_r + \frac{1}{r}\cos\theta\cancel{\cos\varphi}\,\partial_\theta - \frac{\sin\varphi}{r\sin\theta}\,\partial_\varphi\right)\Big\}$$

$$\boxed{= \frac{\hbar}{i}\frac{\partial}{\partial\varphi}}$$

同様に

$$L_x = \frac{\hbar}{i}\left(-\sin\varphi\,\partial_\theta - \frac{\cos\theta}{\sin\theta}\cos\varphi\,\partial_\varphi\right)$$
$$L_y = \frac{\hbar}{i}\left(\cos\varphi\,\partial_\theta - \frac{\cos\theta}{\sin\theta}\sin\varphi\,\partial_\varphi\right)$$

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$$L_x^2+L_y^2+L_z^2 = -\hbar^2\left(-\sin\varphi\,\partial_\theta - \frac{\cos\theta}{\sin\theta}\cos\varphi\,\partial_\varphi\right)^2$$
$$-\hbar^2\left(\cos\varphi\,\partial_\theta - \frac{\cos\theta}{\sin\theta}\sin\varphi\,\partial_\varphi\right)^2$$
$$-\hbar^2\,\partial_\varphi^2$$

$$= -\hbar^2\Big\{\partial_\theta^2 + \underline{\frac{\cos^2\theta}{\sin^2\theta}\,\partial_\varphi^2 + \partial_\varphi^2}$$
$$+\cancel{\sin\varphi\cos\varphi\left(\frac{-\sin\theta}{\sin\theta} - \frac{\cos^2\theta}{\sin^2\theta}\right)}\partial_\varphi$$
$$-\cancel{\sin\varphi\cos\varphi\left(\frac{-\sin\theta}{\sin\theta} - \frac{\cos^2\theta}{\sin^2\theta}\right)}\partial_\varphi$$
$$+\frac{\cos\theta}{\sin\theta}\cos^2\varphi\,\partial_\theta + \frac{\cos\theta}{\sin\theta}\sin^2\varphi\,\partial_\theta\Big\}$$

$$= -\hbar^2\left(\partial_\theta^2 + \frac{\cos\theta}{\sin\theta}\,\partial_\theta + \frac{1}{\sin^2\theta}\,\partial_\varphi^2\right)$$

$$= \ell^2\hbar^2$$

(別の導出法)

2. 角運動量

2.1. 角運動量演算子(の極座標表示)

$$\ell^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

は角運動量の2乗を$\hbar^2$で割ったもの:

$$\mathbb{L}^2 = (\mathbb{r}\times\mathbb{p})^2 = \ell^2\hbar^2$$

(note) $\nabla = \mathbb{e}_r\,\partial_r + \frac{1}{r}\,\mathbb{e}_\theta\,\partial_\theta + \frac{1}{r\sin\theta}\,\mathbb{e}_\varphi\,\partial_\varphi$

(note) $\mathbb{r} = r\,\mathbb{e}_r$

$$\mathbb{L} = \mathbb{r}\times\mathbb{p} = r\,\mathbb{e}_r \times \frac{\hbar}{i}\nabla$$

$$= \frac{\hbar}{i}\cdot\cancel{r}\cdot\left\{\cancel{\mathbb{e}_r\times\mathbb{e}_r\,\partial_r} + \frac{1}{\cancel{r}}(\mathbb{e}_r\times\mathbb{e}_\theta)\,\partial_\theta + \frac{1}{\cancel{r}\sin\theta}(\mathbb{e}_r\times\mathbb{e}_\varphi)\,\partial_\varphi\right\}$$

(note) $\mathbb{e}_r\times\mathbb{e}_\theta = \mathbb{e}_\varphi$
$\mathbb{e}_r\times\mathbb{e}_\varphi = -\mathbb{e}_\theta$

$$\boxed{\mathbb{L} = \frac{\hbar}{i}\left(\mathbb{e}_\varphi\,\partial_\theta - \frac{1}{\sin\theta}\,\mathbb{e}_\theta\,\partial_\varphi\right)}$$

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(note) $\mathbb{e}_\theta = \cos\theta\cos\varphi\,\mathbb{e}_x + \cos\theta\sin\varphi\,\mathbb{e}_y - \sin\theta\,\mathbb{e}_z$

$\mathbb{e}_\varphi = -\sin\varphi\,\mathbb{e}_x + \cos\varphi\,\mathbb{e}_y$

$\downarrow$

$$\mathbb{L} = \underbrace{\frac{\hbar}{i}\left(-\sin\varphi\,\partial_\theta - \frac{\cos\theta}{\sin\theta}\cos\varphi\,\partial_\varphi\right)}_{= L_x}\mathbb{e}_x$$

$$+\underbrace{\frac{\hbar}{i}\left(\cos\varphi\,\partial_\theta - \frac{\cos\theta}{\sin\theta}\sin\varphi\,\partial_\varphi\right)}_{= L_y}\mathbb{e}_y$$

$$+\underbrace{\frac{\hbar}{i}\partial_\varphi}_{= L_z}\mathbb{e}_z$$

$$= L_x\,\mathbb{e}_x + L_y\,\mathbb{e}_y + L_z\,\mathbb{e}_z$$

(note) $L_\pm \equiv L_x \pm i L_y$

$$= \frac{\hbar}{i}\Big\{(-\sin\varphi \pm i\cos\varphi)\,\partial_\theta$$

$$- \frac{\cos\theta}{\sin\theta}(\cos\varphi \pm i\sin\varphi)\,\partial_\varphi\Big\}$$

$$= -\hbar\, e^{\pm i\varphi}\left(\pm\partial_\theta + i\frac{\cos\theta}{\sin\theta}\partial_\varphi\right)$$

$\uparrow$

$$-\sin\varphi \pm i\cos\varphi = -\frac{e^{i\varphi}-e^{-i\varphi}}{2i} \pm i\,\frac{e^{i\varphi}+e^{-i\varphi}}{2}$$

$$= \pm i \cdot e^{\pm i\varphi}$$

$$\cos\varphi \pm i\sin\varphi = \mp i\,(-\sin\varphi \pm i\cos\varphi) = e^{\pm i\varphi}$$