時間に依存する摂動論

$$H = H_0 + V(t)$$

$$H_0 \phi_n = \varepsilon_n \phi_n$$

$$i\hbar \frac{\partial}{\partial t} \Psi(t) = (H_0 + V)\Psi(t)$$

$$\Psi(t=0) = \phi_n$$

$$\Psi(t) = \sum_m c_m(t)\, e^{-i\varepsilon_m t/\hbar}\, \phi_m$$

$\downarrow$

$$i\hbar \dot{c}_k = \sum_m e^{i(\varepsilon_k - \varepsilon_m)t/\hbar} \langle \phi_k | V | \phi_m \rangle c_m$$

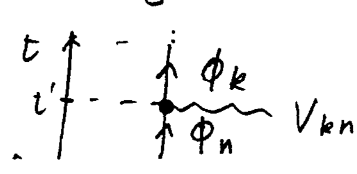
$\downarrow$

$$c_k(t) = \delta_{k,n} + \frac{1}{i\hbar}\int_0^t dt'\, e^{i(\varepsilon_k - \varepsilon_n)t'/\hbar}\, V_{kn}(t')$$

$$+ \cdots$$

$$\begin{cases} P_k(t) = |\langle \phi_k | \Psi(t) \rangle|^2 = |c_k(t)|^2 & (k \neq n) \\ P_n(t) = 1 - \sum_{k \neq n} P_k(t) \end{cases}$$

$$c_k(t)\, e^{-i\varepsilon_k t/\hbar} \approx \frac{1}{i\hbar}\int_0^t dt'\, e^{-i\varepsilon_k (t-t')/\hbar}\, e^{-i\varepsilon_n t'/\hbar}\, V_{kn}(t')$$



∘ 例)

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$$V(t) = \begin{cases} \lambda x & (0 \le t \le T) \\ 0 & (t > T) \end{cases}$$

(note) $V(t) = \lambda x = \lambda \alpha_0 (a + a^+)$; $\alpha_0 = \sqrt{\frac{\hbar}{2m\omega}}$

$\Downarrow$

$$\lambda x |0\rangle = \lambda \alpha_0 |1\rangle$$

$\Downarrow$

$$C_1 = \frac{1}{i\hbar} \int_0^T dt \; e^{i(\hbar\omega - 0)t/\hbar} \cdot \lambda\alpha_0$$

$$= \frac{\lambda\alpha_0}{i\hbar} \int_0^T dt \; e^{i\omega t}$$

$$= \frac{\lambda\alpha_0}{i\hbar} \cdot \frac{1}{i\omega} (e^{i\omega T} - 1)$$

$$= -\frac{\lambda\alpha_0}{\hbar\omega} e^{i\omega T/2} \cdot 2i \sin\left(\frac{\omega T}{2}\right)$$

$\Downarrow$

$$P_1 = |C_1|^2 = \left(\frac{\lambda\alpha_0}{\hbar\omega}\right)^2 \cdot 4\sin^2\left(\frac{\omega T}{2}\right)$$

$$P_k = 0 \qquad (k > 1)$$

$$P_0 = 1 - P_1 = 1 - \left(\frac{\lambda\alpha_0}{\hbar\omega}\right)^2 \cdot 4\sin^2\left(\frac{\omega T}{2}\right)$$

(note) $\int_{-\infty}^{\infty} dx \, \frac{1}{x^2} \sin^2 x = \pi$

2.4 周期的な摂動による遷移.

$$V(t) = V\, e^{\mp i\omega t}$$

$\downarrow$

$$\int_0^t dt' \, e^{i\varepsilon_{kn} t'/\hbar} \, V_{kn}(t') dt' = V_{kn} \int_0^t dt' \, e^{i(\varepsilon_{kn} \mp \hbar\omega) t'/\hbar}$$

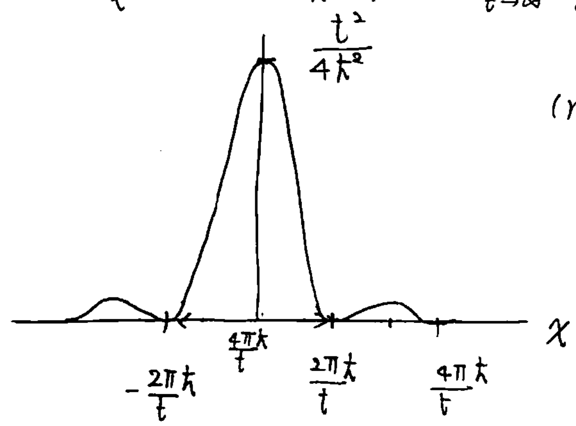
$$= V_{kn} \frac{\hbar}{i(\varepsilon_{kn} \mp \omega)} \left(\underbrace{e^{i(\varepsilon_{kn} \mp \hbar\omega) t/\hbar} - 1}_{= 2i\, e^{\frac{i\Delta t}{2\hbar}} \frac{e^{i\frac{\Delta t}{2\hbar}} - e^{-i\frac{\Delta t}{2\hbar}}}{2i}} \right)$$

$\downarrow$

$$P_k(t) = \frac{4\lambda^2}{\Delta^2} |V_{kn}|^2 \sin^2\left(\frac{\Delta \cdot t}{2\hbar}\right)$$

$$\Delta = \varepsilon_k - \varepsilon_n \mp \hbar\omega$$

$$f(x) = \frac{1}{x^2} \sin^2\left(\frac{xt}{2\hbar}\right) \xrightarrow[t\to\infty]{} \frac{\pi t}{2\hbar} \delta(x)$$

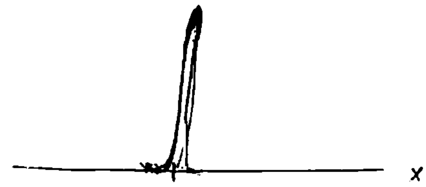


(note) $\int_{-\infty}^{\infty} dx \, \frac{1}{x^2} \sin^2\left(\frac{xt}{2\hbar}\right)$

$$= \frac{t}{2\hbar} \int_{-\infty}^{\infty} dy \, \frac{1}{y^2} \sin^2 y$$

$$= \frac{\pi t}{2\hbar}$$

$\downarrow$



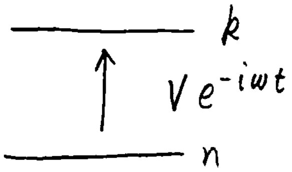
(note)
$\Delta E \cdot \Delta t \sim \hbar$

$\downarrow$

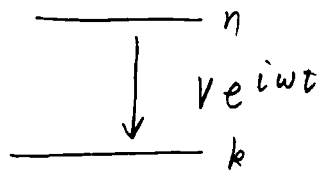
$$\boxed{P_k(t) \longrightarrow \frac{2\pi}{\hbar} t \, \lambda^2 |V_{kn}|^2 \, \delta(\varepsilon_k - \varepsilon_n \mp \hbar\omega)}$$

単位時間当たりの遷移確率：

$$\boxed{\Gamma_k = P_k / t = \frac{2\pi}{\hbar} \lambda^2 |V_{kn}|^2 \, \delta(\varepsilon_k - \varepsilon_n \mp \hbar\omega)}$$



$\varepsilon_k = \varepsilon_n + \hbar\omega$



$\varepsilon_k = \varepsilon_n - \hbar\omega$

。いくつかの状態が ε_k に縮退している時
（終状態が ε_k だけで指定できない時）

（例）3次元の散乱状態　$\varepsilon = \frac{\vec{p}^2}{2m} = \frac{p^2}{2m}$

$$p = \sqrt{2m\varepsilon}$$

$$\begin{pmatrix} p_x = p \\ p_y = p_z = 0 \end{pmatrix} \quad \begin{pmatrix} p_x = p_y = 0 \\ p_z = p \end{pmatrix} \quad \begin{pmatrix} p_x = p_y = \frac{p}{\sqrt{2}} \\ p_z = 0 \end{pmatrix}$$ など

全遷移確率：

$$T = \sum_k T_k = \frac{2\pi}{\hbar} \lambda^2 \sum_k |V_{kn}|^2 \delta(\varepsilon_k - \varepsilon_n \mp \hbar\omega)$$

$$\approx \frac{2\pi}{\hbar} \lambda^2 |V_{fn}|^2 \underbrace{\sum_k \delta(\varepsilon_k - \varepsilon_n \mp \hbar\omega)}_{= \rho(\varepsilon_n \pm \hbar\omega)}$$

（状態密度）
単位エネルギー間隔にある状態の数

$$\boxed{T = \frac{2\pi}{\hbar} |\lambda V_{fn}|^2 \rho(\varepsilon_n \pm \hbar\omega)}$$

Fermi の Golden Rule

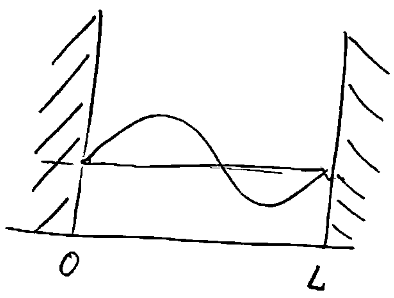
◦例） 終状態が散乱状態の場合（E_f だけ指定）

$$H = \frac{\vec{P}^2}{2m} \qquad \vec{P}:\text{連続量}$$

$$H\psi_f(\vec{x}) = E_f \psi_f(\vec{x})$$

$$\psi_f(\vec{x}) = \psi_{E_x}(x)\,\psi_{E_y}(y)\,\psi_{E_z}(z)$$

$$\left(\frac{p_x^2}{2m} - E_x\right)\psi_{E_x}(x) = 0$$ など。



周期境界条件

$$\psi(0) = \psi(L) = 0$$
$$\psi'(0) = \psi'(L)$$

↓

$$\psi(x) = C\sin kx$$
$$kL = 2n\pi$$

↓

$$k_x = \frac{2\pi}{L}n_x,\quad k_y = \frac{2\pi}{L}n_y,\quad k_z = \frac{2\pi}{L}n_z$$

$$\Gamma_{fi} = \sum_{n_x, n_y, n_z} \frac{2\pi}{\hbar} \lambda^2 |V_{fi}|^2 \delta(\varepsilon_f - \varepsilon_i \mp \hbar\omega)$$

$$\varepsilon_f = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2)$$

$$= \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 (n_x^2 + n_y^2 + n_z^2)$$ を満たすような

(n_x, n_y, n_z) を足す。

$$\sim \int d^3n \, \frac{2\pi}{\hbar} \lambda^2 |V_{fi}|^2 \delta(\varepsilon_f - \varepsilon_i \mp \hbar\omega)$$

$$= \left(\frac{L}{2\pi}\right)^3 \int d^3k \cdot \frac{2\pi}{\hbar} \lambda^2 |V_{fi}|^2 \delta(\varepsilon_f - \varepsilon_i \mp \hbar\omega)$$

$$= \int \underbrace{\boxed{\frac{d^3k}{(2\pi)^3} \cdot L^3}}_{= d^3n} \cdot \frac{2\pi}{\hbar} \lambda^2 |V_{fi}|^2 \delta(\varepsilon_f - \varepsilon_i \mp \hbar\omega)$$

$$\phi_f \sim \frac{1}{\sqrt{L^3}} e^{i\vec{k}\cdot\vec{r}}$$

↓

L^3 はキャンセルされる。