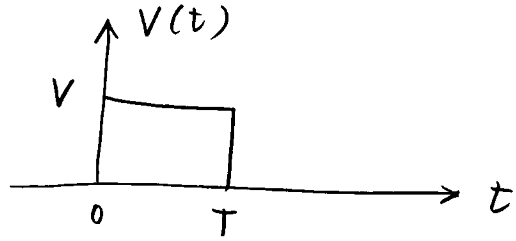
$$e^{i\theta}-1 = e^{\frac{i\theta}{2}}\left(e^{\frac{i\theta}{2}}-e^{-\frac{i\theta}{2}}\right) = e^{\frac{i\theta}{2}}\cdot 2i\sin\left(\frac{\theta}{2}\right)$$

1.3 時間を含まない摂動による遷移

$$V(t) = \hat{V} \quad (0 \le t \le T)$$



$$C_k^{(1)}(T) = \int_0^T \tilde{V}_{kn}(t)\,dt \times \frac{1}{i\hbar}$$

$$= V_{kn}\int_0^T e^{i(E_k-E_n)t/\hbar}\,dt \times \frac{1}{i\hbar}$$

$$= V_{kn}\frac{\hbar}{i(E_k-E_n)}\left(e^{i(E_k-E_n)T/\hbar}-1\right)\times\frac{1}{i\hbar}$$

$$= V_{kn}\frac{\hbar}{iE_{kn}}e^{iE_{kn}T/2\hbar}\left(e^{iE_{kn}T/2\hbar}-e^{-iE_{kn}T/2\hbar}\right)\times\frac{1}{i\hbar}$$

$$E_{kn} \equiv E_k - E_n$$

$$= -2i\frac{V_{kn}}{E_{kn}}e^{iE_{kn}T/2\hbar}\sin\left(\frac{E_{kn}T}{2\hbar}\right)$$

$$P_k(t) = \left(\frac{V_{kn}}{E_{kn}}\right)^2\cdot 4\sin^2\left(\frac{E_{kn}}{2\hbar}T\right) \xrightarrow[T\to\infty]{} \frac{2\pi}{\hbar}T|V_{kn}|^2 \times \delta(E_k-E_n)$$

- 収束の条件（高次の摂動項が小さくなるための条件）

$$|V_{kn}| \ll |E_{kn}|$$

又は

$$|E_{kn}T/2\hbar| \ll 1$$

典型的な遷移時間に比べて十分長い時間をかけて観測 ただし摂動論が成り立つくらいに十分小さく

(複習) 周期的な摂動による遷移

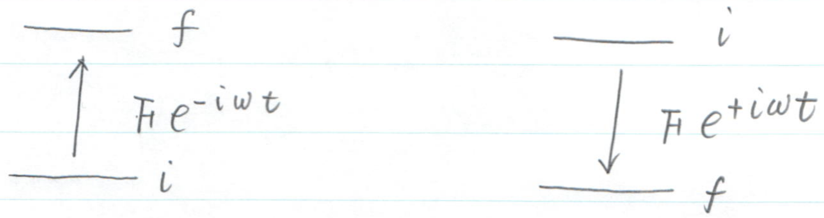
$$V(t) = \hat{F} e^{\pm i\omega t}$$

↓ 単位時間当りの遷移確率：

$$W_{fi} = \frac{2\pi}{\hbar} |\langle f|\hat{F}|i\rangle|^2 \cdot \underbrace{\sum_f \delta(E_f - E_i \pm \hbar\omega)}_{= \rho(E_f = E_i \mp \hbar\omega)}$$

$E = E_f$ を持つ終状態を全て足す

"Fermi の Golden Rule (黄金則)"



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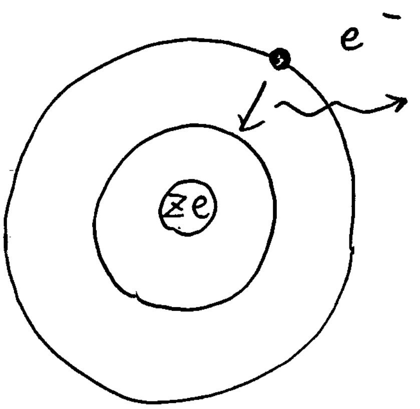
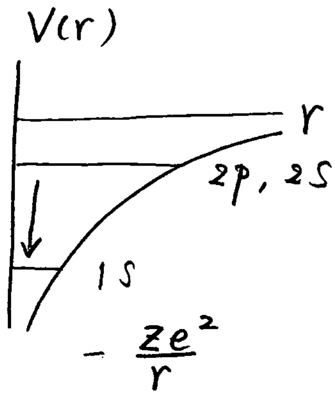
1.5 電磁遷移

原子と電磁場の相互作用





$$H=\frac{\mathbb{P}^2}{2m}+V(r)$$

cf. 量子Ⅱ
ゼーマン効果

$$\longrightarrow \frac{1}{2m}\left(\mathbb{P}+\frac{e}{c}A(\mathbb{r},t)\right)^2+V(r)-e\phi(\mathbb{r},t)$$

と変更すると電子と電磁場の相互作用を記述できる。

(note) 古典的運動方程式

$$m\ddot{\mathbb{r}}=-e\left[\mathbb{E}(\mathbb{r},t)+\frac{1}{c}\mathbb{v}\times\mathbb{B}(\mathbb{r},t)\right]$$

が導びかれる。

$$\begin{cases}\mathbb{E}=-\nabla\phi-\frac{1}{c}\frac{\partial}{\partial t}A\\ \mathbb{B}=\nabla\times A\end{cases}$$

$$H=\frac{1}{2m}\mathbb{P}^2+V(r)+\frac{e}{2mc}(\mathbb{P}\cdot A+A\cdot\mathbb{P})+\underbrace{\frac{e^2}{2mc^2}A^2}_{=\,O(e^2)\,\to\,\text{ムシ}}-e\phi$$

クーロン・ゲージ $\nabla\cdot A=0$, $\phi=0$ をとる ←

$$\downarrow$$

$$H=\frac{1}{2m}\mathbb{P}^2+V(r)+\boxed{\frac{e}{mc}A\cdot\mathbb{P}}$$

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$$H = \underbrace{\frac{p^2}{2m} + V(r)}_{=\,H_0} + \underbrace{\boxed{\frac{e}{mc} A\cdot p}}_{=\,\text{摂動}\,:\,V(r,t)}$$

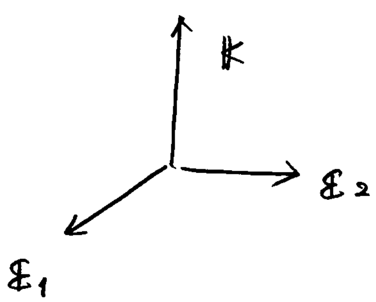
(準備) マクスウェル方程式: $\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) A(r,t) = 0$

解: $A(r,t) = A_0\, e^{i(k\cdot r - \omega t)} + c.c.$

$(\omega = ck)$

クーロン・ゲージ $\nabla \cdot A = 0$

$\rightarrow \quad k \cdot A_0 = 0$ (横波条件)



2つの独立解

$A_0 = A_0 \cdot \varepsilon_\alpha$ $(\alpha = 1, 2)$

偏極ベクトル

$A_0 = \sqrt{\frac{2\pi c^2 \hbar}{\omega}}\, a_{k\alpha}$ (←量子電気力学)

フォトン消滅演算子

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$$\frac{2\pi}{\hbar}\cdot\frac{1}{(2\pi)^3}\cdot\frac{k^2}{\hbar c}\cdot e^2\cdot \underbrace{\frac{2\pi c^2\hbar}{\omega}}_{(A_0^*)^2} = \frac{1}{2\pi}\cdot\frac{e^2}{\hbar c}\cdot\cancel{\frac{(ck)^2}{\omega}}\cdot\frac{1}{2\pi}\cdot\frac{e^2}{\hbar c}\cdot\frac{\hbar\omega}{\hbar}$$

○ 電磁波の放出

$$H_{int} = \frac{e}{mc}\,A_0^*\,\varepsilon_\alpha\cdot \mathbb{P}\,e^{-i(k\cdot r-\omega t)}$$

$$\begin{array}{c} \text{―― } n \\ H_{int}\downarrow \rightsquigarrow \hbar\omega \\ \text{―― } k \end{array}$$

始状態　$|\phi_n\rangle$

終状態　$|\phi_k\rangle$　$\left(+\ \text{フォトン}\ |\mathbb{k}\underset{\uparrow\,\text{偏極}}{\alpha}\rangle\right)$

$$W_{fi} = \frac{2\pi}{\hbar}\int \overbrace{\frac{d^3k}{(2\pi)^3}\sum_\alpha}^{\text{終状態の和}}\ \overbrace{\delta(E_k-E_n+\hbar\omega)}^{\text{エネルギー保存則}}$$

$$\times\left|\langle\phi_k|\frac{e}{mc}A_0^*\,\varepsilon_\alpha\cdot\mathbb{P}\,e^{-ik\cdot r}|\phi_n\rangle\right|^2$$

(note)　$d^3k = k^2dk\,d\Omega_k = k^2\cdot\frac{d(\overset{\omega}{ck})}{c}d\Omega_k$

$$= \frac{k^2}{\hbar c}\,d(\hbar\omega)\,d\Omega_k$$

$\downarrow$

$$W_{fi} = \sum_\alpha\int \underbrace{d\Omega_k}\ \frac{1}{2\pi}\cdot\frac{e^2}{\hbar c}\cdot\frac{E_n-E_k}{\hbar}$$

$$\times\left|\frac{1}{mc}\ \langle\phi_k|\,e^{-ik\cdot r}\ \varepsilon_\alpha\cdot\mathbb{P}\,|\phi_n\rangle\right|^2$$

$\langle\mathbb{k}\alpha|a^+_{k\alpha}|0\rangle$
$=1$

フォトンの放出方向

$$\frac{dW_{fi}}{d\Omega_k} = \sum_\alpha\ \frac{1}{2\pi}\cdot\frac{e^2}{\hbar c}\,\frac{E_n-E_k}{\hbar}\ \left|\frac{1}{mc}\ \langle\phi_k|\cdots|\phi_n\rangle\right|^2$$

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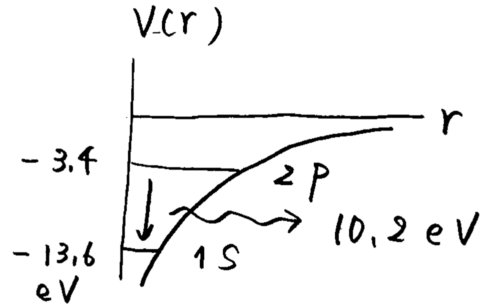
• 双極子近似

$$e^{-i\mathbf{k}\cdot\mathbf{r}} \sim 1 \qquad (\mathbf{k}\cdot\mathbf{r} \ll 1)$$

$$k \ll \frac{1}{r} \qquad (\text{長波長近似})$$

水素原子 : $E_n = -\frac{1}{2} m c^2 \cdot \frac{\alpha^2}{n^2} \qquad (\alpha = \frac{1}{137})$

$$\downarrow \quad E_2 - E_1 \sim 10.2\,\mathrm{eV}$$



$$\hbar\omega \sim 10\ \mathrm{eV}$$

$$k = \frac{\omega}{c} = \frac{\hbar\omega}{\hbar c} \sim \frac{10\ \mathrm{eV}}{2000\ \mathrm{eV}\cdot\mathring{A}} = \frac{1}{200}\ \mathring{A}^{-1}$$

$$r \sim O(\mathring{A})$$

$$\Downarrow$$

$$k \ll \frac{1}{r}.$$

(note) $\hbar c = 197.3\ \mathrm{MeV}\cdot\mathrm{fm}$ ($\mathrm{fm} = 10^{-15}\,\mathrm{m}$)

$$= 1973\ \mathrm{eV}\cdot\mathring{A}$$

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$\downarrow$

$$\langle \phi_k | e^{-i\mathbf{k}\cdot\mathbf{r}}\, \boldsymbol{\varepsilon}_\alpha \cdot \mathbf{p} | \phi_n \rangle \sim \boldsymbol{\varepsilon}_\alpha \cdot \langle \phi_k | \mathbf{p} | \phi_n \rangle$$

(note)

$$[\mathbf{p}^2, \mathbf{r}] = -[\mathbf{r}, \mathbf{p}^2] = -[\underbrace{i\hbar \nabla_{\mathbf{p}}}_{[\mathbf{r},\mathbf{p}]=i\hbar}, \mathbf{p}^2]$$
$$= -2i\hbar\, \mathbf{p}$$

$\Downarrow$

$$\langle \phi_k | \mathbf{p} | \phi_n \rangle = \langle \phi_k | \frac{1}{-2i\hbar} [\mathbf{p}^2, \mathbf{r}] | \phi_n \rangle$$

$$= \langle \phi_k | \frac{2m}{-2i\hbar} [\underbrace{\frac{\mathbf{p}^2}{2m} + V(r)}_{H_0}, \mathbf{r}] | \phi_n \rangle$$

$$= \frac{im}{\hbar} \langle \phi_k | H_0 \mathbf{r} - \mathbf{r} H_0 | \phi_n \rangle$$

$$= \frac{im}{\hbar} (E_k - E_n) \langle \phi_k | \mathbf{r} | \phi_n \rangle$$

E1 遷移
(電気双極子)遷移