[復習]

$$\psi(\mathbf{r}) \to \frac{i}{2k}\sum_{\ell}(2\ell+1)\, i^{\ell}\cdot\frac{1}{r}\left[e^{-i(kr-\frac{\ell\pi}{2})} - S_\ell\, e^{i(kr-\frac{\ell\pi}{2})}\right]P_\ell(c$$

$$= e^{i\mathbf{k}\cdot\mathbf{r}} + \underbrace{f(\theta)\frac{e^{ikr}}{r}}$$

$$f(\theta) = \sum_{\ell}(2\ell+1)\frac{S_\ell - 1}{2ik}P_\ell(\cos\theta)$$

散乱波のフラックス：

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

全断面積 $\sigma_{tot} = \int \frac{d\sigma}{d\Omega}d\Omega$

$$= \frac{\pi}{k^2}\sum_{\ell}|S_\ell - 1|^2$$

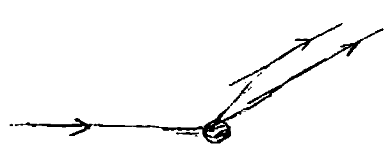
◦ 吸収のある時 ： $|S_\ell| < 1$

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V - iW\right)\psi = E\psi$$

◦ 吸収断面積

反応プロセス
- 弾性散乱
- 非弾性散乱
- 粒子移行
- 複合粒子形成（核融合）

⋮



弾性フラックスの減少
（吸収）

光学ポテンシャル

$$V_{opt}(r) = V(r) - iW(r) \qquad (W > 0)$$

(note)

$$\nabla\cdot\boldsymbol{j} = \frac{\hbar}{2im}\left(\cancel{\nabla\psi^*\cdot\nabla\psi} + \psi^*\nabla^2\psi - \cancel{\nabla\psi\cdot\nabla\psi^*} - \psi\nabla^2\psi^*\right)$$

（フラックスの発散）

$$= \frac{\hbar}{2im}\left\{\psi^*\left(-\frac{2m}{\hbar^2}\right)(E-V+iW)\psi - \psi\left(-\frac{2m}{\hbar^2}\right)(E-V-iW)\psi^*\right\}$$

$$= -\frac{2}{\hbar}|\psi|^2 W = -\frac{2}{\hbar}\rho W$$

フラックスの減少

$$\leftrightarrow \quad |S_l| < 1$$

(note) ガウスの定理

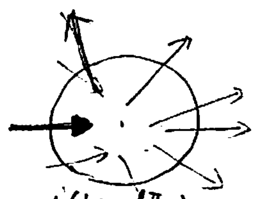
$$\int_S \vec{j}\cdot\vec{n}\,dS = \int_V \nabla\cdot\boldsymbol{j}\,dV$$

11/24

$$\vec{j} \sim \frac{\hbar}{2im}\left(\psi^* e_r \partial_r \psi - c.c.\right)$$

$$\psi(r) \to \frac{i}{2k}\sum_l (2l+1)\, i^l \left[\underline{\frac{e^{-i(kr-\frac{l\pi}{2})}}{r}} - S_l \underline{\frac{e^{i(kr-\frac{l\pi}{2})}}{r}}\right] P_l(\cos\theta)$$

全内向フラックス：



$$\psi_{in}(r) = \frac{i}{2k}\sum_l (2l+1)\, i^l\, \frac{e^{-i(kr-\frac{l\pi}{2})}}{r} P_l(\cos\theta)$$

$$= \frac{i}{2kr}\underline{\sum_l (2l+1)\cdot(-)^l}\; e^{-ikr}\, \underline{P_l(\cos\theta)}$$

$$j_{in} = \frac{k\hbar}{m}\cdot\frac{1}{r^2}\left|\frac{i}{2k}\sum_l (2l+1)(-)^l P_l(\cos\theta)\right|^2 e_r$$

$$\downarrow$$

$$j_{in}^{net} = \int r^2 d\Omega\; j_{in} = \frac{k\hbar}{m}\,\frac{1}{\cancel{4}k^2}\sum_l (2l+1)\cdot\cancel{4}\pi$$

全外向きフラックス

$$\psi_{out} = \frac{i}{2k}\sum_l (2l+1)\, i^l\, S_l \frac{e^{i(kr-\frac{l\pi}{2})}}{r} P_l(\cos\theta)$$

$$j_{out}^{net} = \frac{k\hbar}{m}\cdot\frac{\pi}{k^2}\sum_l (2l+1)|S_l|^2$$

$$\downarrow$$

吸収断面積： $\boxed{\sigma_{abs} = \frac{j_{in} - j_{out}}{j_{inc.}} = \frac{\pi}{k^2}\sum_l (2l+1)(1-|S_l|^2)}$

(note) if $|S_l| = 1$, $\sigma_{abs} = 0$.

(note) $\sigma_{el} = \frac{\pi}{k^2} \sum_{\ell} (2\ell+1) |1 - S_\ell|^2$

$\downarrow$

$$\sigma_{tot} = \sigma_{el} + \sigma_{abs}$$

$$= \frac{\pi}{k^2} \sum_{\ell} (2\ell+1) \left\{ 1 \cancel{- |S_\ell|^2} + 1 - (S_\ell + S_\ell^*) \cancel{+ |S_\ell|^2} \right\}$$

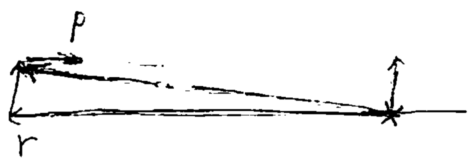
$$= \frac{2\pi}{k^2} \sum_{\ell} (2\ell+1) \left(1 - \frac{S_\ell + S_\ell^*}{2}\right)$$

(note)

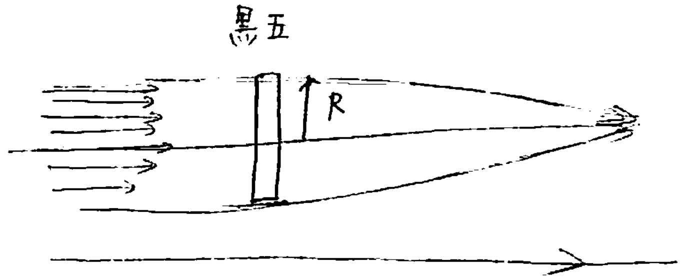
$$\mathrm{Im}\, f(\theta=0) = \mathrm{Im} \sum_{\ell} (2\ell+1) \frac{S_\ell - 1}{2ik}$$

$$= - \sum_{\ell} (2\ell+1) \frac{\mathrm{Re}[S_\ell] - 1}{2k} = \frac{1}{2k} \sum_{\ell} (2\ell+1) \left(1 - \frac{S_\ell + S_\ell^*}{2}\right)$$

$$= \frac{k}{4\pi} \sigma_{tot}$$



∘ 陰散乱



$L = kb$ impact parameter
$\overset{\shortparallel}{\vec{r}\times\vec{p}}$

$$S_\ell = \begin{cases} 0 & \ell \lesssim kR \\ 1 & \ell > kR \end{cases} \quad （全吸収）$$

$$\sigma_{abs} = \frac{\pi}{k^2}\sum_{\ell=0}^{kR}(2\ell+1)\underbrace{(1-|S_\ell|^2)}_{\shortparallel\atop 1} = \frac{\pi}{k^2}\cdot(kR)^2 = \pi R^2$$

$\updownarrow$
geometric region

$$\sigma_{el} = \frac{\pi}{k^2}\sum_\ell (2\ell+1)\,|1-S_\ell|^2$$
$$= \frac{\pi}{k^2}\sum_{\ell=0}^{kR}(2\ell+1) = \pi R^2$$

弾性散乱：ディスクの端での回折
$\leftrightarrow$ geometric radius $= R$

$$f(\theta) = \sum_l (2l+1) \frac{S_l - 1}{2ik} P_l(\cos\theta)$$

$$= -\sum_{l=0}^{kR} \underline{(2l+1)} \cdot \frac{1}{2ik} \underline{P_l(\cos\theta)}$$

$f(\theta)$: strongly peaked at $\theta = 0$ (焦点現象)

(note) $P_l(1) = 1$ for any l

$\downarrow$

$$P_l(\cos\theta) \approx J_0(l\theta) \quad \text{for } \theta \ll 1,\ l \gg 1$$

$\downarrow$

$$f(\theta) \sim \frac{i}{2k} \sum_l \underline{(2l+1)} J_0(l\theta)$$

$\int dl\,(2l+1) \sim 2k^2 \int b\,db$

$$\underset{l=kb}{\sim} ik \int_0^R \underbrace{b\,db} J_0(kb\theta)$$

$$\underset{\int x J_0(x) dx = x J_1(x)}{=} \frac{iR}{\theta} \underbrace{J_1(kR\theta)}$$ (フラウンホーファー振幅)

$\downarrow$

$$\Delta\theta \sim \frac{\pi}{kR}$$

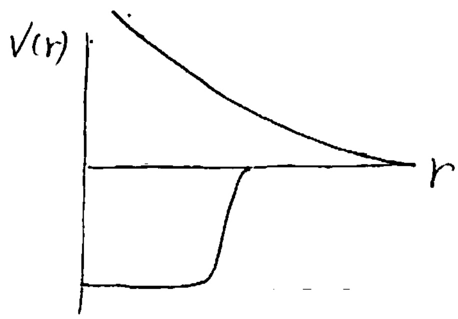
cf. 軽イオン反応 : $^{16}O + {}^{12}C$ @ 168 MeV

$E_{lab}(^{16}O) =$

5.4. 低エネルギー散乱

○ 一般的な考察

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+\frac{l(l+1)\hbar^2}{2\mu r^2}+V(r)-E\right]u_l(r)=0$$



as $l\to\infty$

$$\frac{l(l+1)\hbar^2}{2\mu r^2} \gg V(r)$$

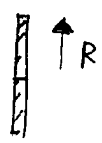
↑ less important

cf. ボルン近似

⇕

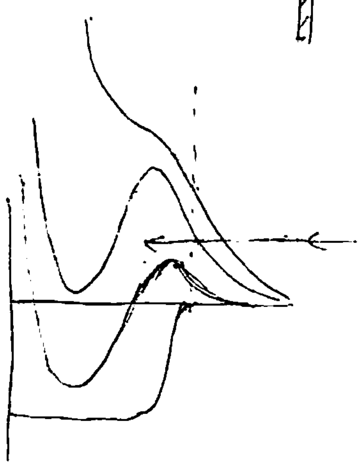
$\delta_l \to 0$

(note) black disk



$L_{max} = kR$

(note)

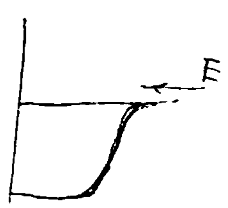


反応が起きるためには interaction range までトンネルする必要あり

(note) $E \to 0$ only $l=0$ のみ寄与

↓

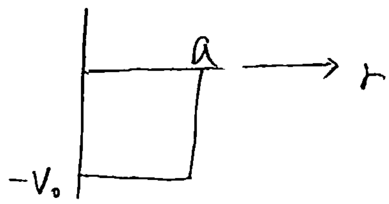
遠心力バリア $=0$



。 $E \to 0$ での振るまい（しきい値：threshold の振るまい）

（例）square well potential

$$\psi(r) = R_\ell(r)\, Y_{\ell m}(\hat{r})$$



$r < a$

$$R_\ell(r) = A\, j_\ell(Kr)$$

$r \geq a$

$$R_\ell(r) = B\, j_\ell(kr) + C\, n_\ell(kr)$$

$$K = \sqrt{\frac{2m}{\hbar^2}(E+V_0)}, \quad k = \sqrt{\frac{2m}{\hbar^2}E}$$

(note)

$$R_\ell(r) \to \left[B \sin\left(kr - \frac{\ell\pi}{2}\right) - C\cos\left(kr - \frac{\ell\pi}{2}\right)\right]\cdot\frac{1}{kr}$$

$$\Leftrightarrow \frac{1}{kr}\sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right) = \frac{1}{kr}\left[\sin\left(kr - \frac{\ell\pi}{2}\right)\cos\delta_\ell + \cos\left(kr - \frac{\ell\pi}{2}\right)\sin\delta_\ell\right]$$

$$\downarrow \quad \boxed{\tan\delta_\ell = -\frac{C}{B}}$$

matching at $r = a$:

$$\frac{R_\ell'}{R_\ell} = \frac{K j_\ell'(Ka)}{j_\ell(Ka)} = k\,\frac{j_\ell'(ka) + \frac{C}{B} n_\ell'(ka)}{j_\ell(ka) + \frac{C}{B} n_\ell(ka)}$$

$$\downarrow$$

$$\tan\delta_\ell = -\frac{C}{B} = \frac{k j_\ell'(ka)\, j_\ell(Ka) - K j_\ell'(Ka)\, j_\ell(ka)}{k n_\ell'(ka)\, j_\ell(Ka) - K n_\ell(ka)\, j_\ell'(Ka)}$$

$ka \ll l$ の極限：

$$j_l(ka) \sim \frac{(ka)^l}{(2l+1)!!}\ , \qquad n_l(ka) \sim -\frac{(2l-1)!!}{(ka)^{l+1}}$$

$\downarrow$

$$\tan\delta_l \sim \frac{k\cdot\dfrac{l(ka)^{l-1}}{(2l+1)!!}\, j_l(\kappa a) - \kappa\cdot\dfrac{(ka)^l}{(2l+1)!!}\, j_l'(\kappa a)}{k\cdot(l+1)\dfrac{(2l-1)!!}{(ka)^{l+2}}\cdot j_l(\kappa a) + \kappa\cdot\dfrac{(2l-1)!!}{(ka)^{l+1}}\, j_l'(\kappa a)}$$

$$= \frac{2l+1}{[(2l+1)!!]^2}\cdot(ka)^{2l+1}\ \frac{\cancel{k}l\ j_l(\kappa a) - \kappa\cdot\cancel{k}a\ j_l'(\kappa a)}{\cancel{k}(l+1)\, j_l(\kappa a) + \cancel{k}\kappa a\ j_l'(\kappa a)}$$

$$\equiv -C_l\ k^{2l+1}$$

(square well potential 以外でも成り立つ)

$\downarrow$

$$\boxed{\tan\delta_l \sim \delta_l \sim -C_l\, k^{2l+1}} \qquad (k \to 0)$$

(note)

as $\quad k \to 0$

$$\delta_{l=0} \gg \delta_{l=1} \gg \delta_{l=2} \ ----$$

(note)

$$\sigma_{tot} = \frac{4\pi}{k^2}\sum_l (2l+1)\sin^2\delta_l \sim \frac{4\pi}{k^2}\sum_l (2l+1)\,\delta_l^2$$

$$\sim \frac{4\pi}{k^2}\sum_l (2l+1)\, C_l^2\, k^{4l+2}$$

$$\longrightarrow 4\pi\, C_{l=0}^2 \qquad (k\to 0)$$

・散乱長

$$\tan \delta_\ell = -c_\ell\, k^{2\ell+1}$$

$\ell = 0$:

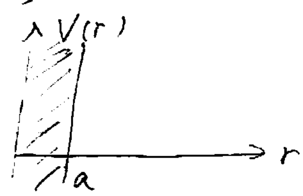
$$k \cot \delta_\ell = -\frac{1}{c_\ell} \equiv -\frac{1}{a}$$

散乱長

$$\sigma_{tot} = 4\pi a^2$$

(note) 剛体球による散乱 (s-wave)

$$V(r) = \begin{cases} \infty & (r < a) \\ 0 & (r > a) \end{cases}$$



$$u(r) = \sin(kr + \delta_0)$$

$$u(r=a) = 0 \quad \Rightarrow \quad \boxed{\delta_0 = -ka}$$

$$j_0(x) = \frac{\sin x}{x}$$
$$j_0'(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2}$$

・散乱長の意味

square well ($l=0$):

$$\tan\delta_0 = kR \cdot \frac{-\kappa R\, j_0'(\kappa R)}{j_0(\kappa R) + \kappa R\, j_0'(\kappa R)}$$

$$= kR \cdot \frac{-\kappa R \left(\frac{\cos\kappa R}{\kappa R} - \frac{\sin\kappa R}{\kappa^2 R^2}\right)}{\cancel{\frac{\sin\kappa R}{\kappa R}} + \cancel{\kappa R}\left(\frac{\cos\kappa R}{\cancel{\kappa R}} - \cancel{\frac{\sin\kappa R}{\kappa^2 R^2}}\right)}$$

$$= kR\left(\frac{\tan\kappa R}{\kappa R} - 1\right)$$

↓

$$\boxed{a = R\left(1 - \frac{\tan\kappa R}{\kappa R}\right)}$$

(note)

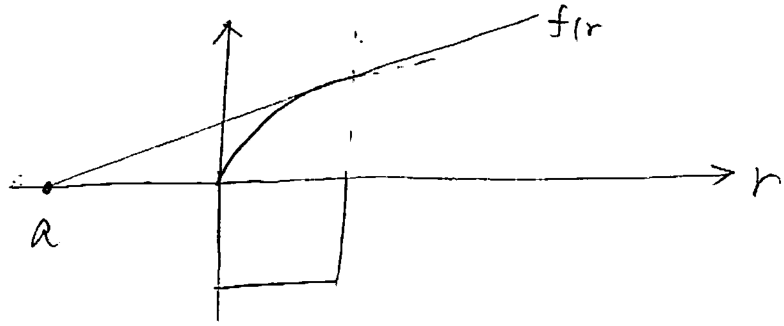
$$u_0(r) = A\sin\kappa r \qquad (r<R)$$

$$f(r) \equiv u_0(R) + u_0'(R)(r-R)$$

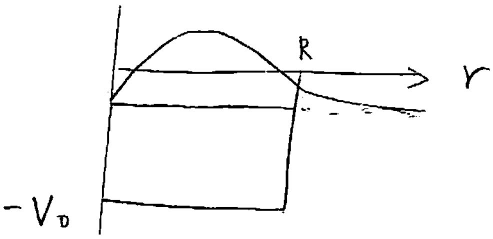
$$= A\left(\sin\kappa R + \kappa\cos\kappa R\cdot(r-R)\right)$$

↓

$$f(r=a) = A\left(\sin\kappa R + \kappa\cos\kappa R\cdot\left(\cancel{R} - \frac{\tan\kappa R}{\kappa} - \cancel{R}\right)\right) = 0$$



(note) 束縛状態からのアプローチ　($E<0$)



$r<R$

$$u(r) = A\sin k'r$$

$r>R$

$$u(r) = Be^{-\tilde{\kappa}r}$$

$$k' = \sqrt{\frac{2m}{\hbar^2}(E+V_0)}\ ,\quad \tilde{\kappa} = \sqrt{\frac{2m}{\hbar^2}|E|}$$

matching at $r=R$:

$$\frac{k'\cos k'R}{\sin k'R} = \frac{-\tilde{\kappa}e^{-\tilde{\kappa}R}}{e^{-\tilde{\kappa}R}}$$

$\Downarrow$

$$\boxed{k'\cot k'R = -\tilde{\kappa}} \quad \leftrightarrow -\frac{1}{a}$$

$\Downarrow$

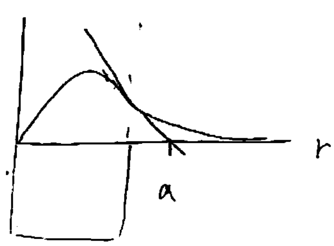
$$a = R\left(1-\frac{\tan kR}{kR}\right) \approx R\left(1-\frac{\tan k'R}{k'R}\right)$$

$$= R+\frac{1}{\tilde{\kappa}} \approx \frac{1}{\tilde{\kappa}}$$

(zero range limit : $R\to 0$)

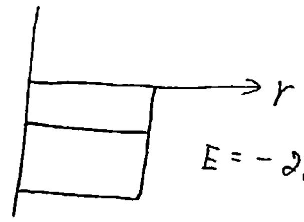
$\Downarrow$ 弱束縛状態がある場合は, $a>0$

ない場合は　$a<0$



・陽子 - 中性子 散乱

重陽子



$S = 1$. (spin triplet)

↓

$$a = \frac{1}{\tilde{k}} = \sqrt{\frac{\hbar^2}{2|m|E}} = \sqrt{\frac{\hbar^2 c^2}{\cancel{2} \cdot \frac{m_N c^2}{\cancel{2}}|E|}}$$

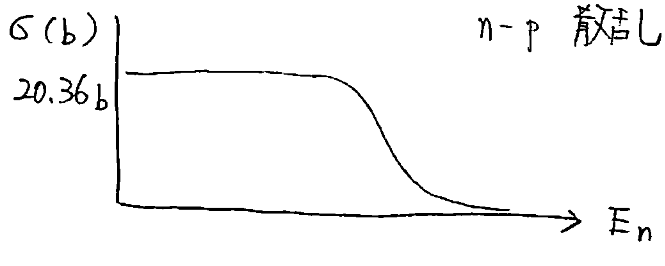
$$= \sqrt{\frac{197^2}{940 \times 2.23}} = 4.31 \text{ fm}$$

$1 b = 10^{-24} cm^2$
$1 fm^2 = 10^{-26} cm^2$
$= 10^{-2} b$

↓

$$\sigma = 4\pi a^2 = 233 \text{ fm}^2 = 2.33 \text{ b}.$$

実験：



n-p 散乱 ← スピン偏極なし

$$\sigma = \frac{3}{4}\sigma_t + \frac{1}{4}\sigma_s = \frac{3}{4} \times 2.33 + \frac{1}{4}\sigma_s$$

↓

$$\sigma_s = 74.45 \text{ b} \quad \rightarrow \quad a_s = \pm 23 \text{ fm}.$$

高精度
実験データ解析 ↓

$$\begin{cases} a_t = 5.415 \pm 0.012 \text{ fm} & r_t = 1.704 \pm 0.028 \text{ fm} \\ a_s = -23.806 \pm 0.028 \text{ fm} & r_s = 2.49 \pm 0.24 \text{ fm} \end{cases}$$

$$\delta(k) = -\delta(-k)$$

(note)

$$\varphi(0,r) \sim \frac{\sin\delta}{\sin\delta} + \frac{k\cos\delta}{\sin\delta}\cdot r = 1 - \frac{r}{a}$$

(note)

$$W[\varphi(k,r)\,\varphi(0,r)]_{r=0} = \underbrace{\varphi(k,0)}_{=1}\underbrace{\varphi'(0,0)}_{=-\frac{1}{a}} - \underbrace{\varphi'(k,0)}_{=k\cot\delta}\underbrace{\varphi(0,0)}_{=1}$$

$$= -\frac{1}{a} - k\cot\delta.$$

$$W[\psi(k,r)\,\psi(0,r)]_{r=0} = 0$$

(note)

$$\psi(k,r) \to \varphi(k,r) \qquad (r\to\infty)$$

$\Downarrow$

$$\int_0^\infty dr\,\frac{d}{dr}\left\{ W[\varphi(k,r)\,\varphi(0,r)] - W[\psi(k,r)\,\psi(0,r)] \right\}$$

$$= -W[\varphi(k,r)\,\varphi(0,r)]_{r=0} = +\frac{1}{a} + k\cot\delta$$

$$= k^2 \int_0^\infty dr\,\left(\varphi(k,r)\varphi(0,r) - \psi(k,r)\psi(0,r)\right)$$

$\Downarrow$

$$k\cot\delta = -\frac{1}{a} + k^2\int_0^\infty dr\,\left(\varphi(k,r)\varphi(0,r) - \psi(k,r)\psi(0,r)\right)$$

$$\sim -\frac{1}{a} + k^2\int_0^\infty dr\,\left(\varphi(0,r)^2 - \psi(0,r)^2\right)$$

$$\equiv -\frac{1}{a} + \frac{1}{2}k^2 \underset{\sim}{r_0} \quad \text{effective range}$$

↓

低エネルギー散乱は　a, r_0 の2個のパラメーター
で記述でき、ポテンシャルの詳細によらない。

↓

$V(r)$ の座標依存性をギロンするためには高エネルギー
散乱が必要.

∘ effective interaction:

$$k \cot \delta \sim -\frac{1}{a} + \frac{1}{2} k^2 r_0 + \cdots$$

低エネルギーではポテンシャルの詳細は重要ではない。

$$\rightsquigarrow \quad V_{eff}(\mathbf{r}) = \frac{2\pi \hbar^2}{m_r} a \, \delta(\mathbf{r})$$

reduced mass

係数は散乱長が a になるように決定。

cf. BEC（アルカリ原子の希薄気体）

cf、H. Esbensen et al. PRC 56('97) 3054

Fetter - Walecka
Eqs. (11.14), (11.53)

原子	$1\,\text{Å} = 10^{-8}$ cm	eV
原子核	$1\,\text{fm} = 10^{-13}$ cm	MeV
素粒子	$\lesssim 10^{-16}$ cm	> GeV

$$j_0(x) = \frac{\sin x}{x} \qquad n_0(x) = -\frac{\cos x}{x}$$

$$j_0'(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2} \qquad n_0'(x) = \frac{\sin x}{x} + \frac{\cos x}{x^2}$$

- Levinson の定理

$$\tan\delta_0 = \frac{k\, j_0'(kR)\, j_0(KR) - K\, j_0'(KR)\, j_0(kR)}{k\, n_0'(kR)\, j_0(KR) - K\, n_0(kR)\, j_0'(KR)}$$

$$= \frac{k\left(\frac{\cos kR}{kR} - \cancel{\frac{\sin kR}{k^2R^2}}\right)\frac{\sin KR}{KR} - K\left(\frac{\cos KR}{KR} - \cancel{\frac{\sin KR}{K^2R^2}}\right)\frac{\sin kR}{kR}}{k\left(\frac{\sin kR}{kR} + \cancel{\frac{\cos kR}{k^2R^2}}\right)\frac{\sin KR}{KR} + K\frac{\cos kR}{kR}\left(\frac{\cos KR}{KR} - \cancel{\frac{\sin KR}{K^2R^2}}\right)}$$

$$= \frac{k\tan KR - K\tan kR}{K + k\tan kR\tan KR}$$

$$k = \sqrt{\frac{2m}{\hbar^2}E}\ , \quad K = \sqrt{\frac{2m}{\hbar^2}(E+V_0)}\ , \quad K_0 \equiv \sqrt{\frac{2m}{\hbar^2}V_0}$$

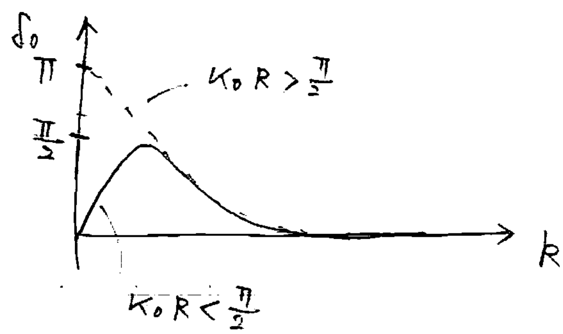
- $K/k \sim 1 \quad (E \to \infty)$: $\tan\delta_0 = 0$

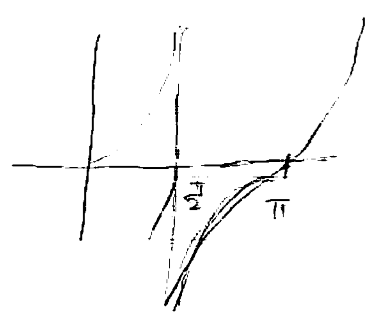
$\Downarrow$

$\delta_0(\infty) = 0.$

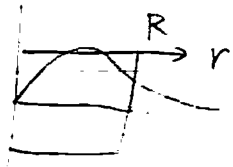
- $E \to 0\ (k \to 0)$ $\quad\Downarrow\quad$ $\tan\delta_0 \to \frac{k}{K_0}\tan K_0R - kR$

if $K_0R \ll 1$	$\tan\delta_0 \to 0+$	$\Downarrow$	$\delta_0 \to 0$
$K_0R = \frac{\pi}{2}$	$\tan\delta_0 \to \infty$	$\Downarrow$	$\delta_0 \to \frac{\pi}{2}$
$K_0R = \frac{\pi}{2}+$	$\tan\delta_0 \to 0-$	$\Downarrow$	$\delta_0 \to \pi$





(note) $E<0$



$r<R$

$$u(r) = A\sin\kappa r$$

$r>R$

$$u(r) = B e^{-\tilde{\kappa} r}$$

$$\tilde{\kappa} = \sqrt{\frac{2m}{\hbar^2}|E|}$$

matching: $\kappa \underbrace{\cot\kappa R}_{\frac{\cos\kappa R}{\sin\kappa R}} = -\tilde{\kappa}$

$\Downarrow$ bound state が存在するための条件： $\kappa R > \frac{\pi}{2}$

$\Downarrow$

bound state が存在する時： $\delta_0(0) = \pi$

ない時： $\delta_0(0) = 0$.

一般に $\boxed{\delta_\ell(0) - \delta_\ell(\infty) = N_B \pi}$ N_B : # of bound state

Levinson's theorem