Heavy-Ion Fusion Reactions around the Coulomb Barrier

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cf. Experimental aspects of H.I. Fusion reactions: lectures by Prof. Mahananda Dasgupta (ANU)
3.11 earthquake

after 1 month

Sendai
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✧ Fusion reactions and quantum tunneling
✧ Basics of the Coupled-channels method
✧ Concept of Fusion barrier distribution
✧ Quasi-elastic scattering and quantum reflection

cf. Experimental aspects of H.I. Fusion reactions: lectures by Prof. Mahananda Dasgupta (ANU)
Fusion: compound nucleus formation

\[ A_{\text{CN}} = A_p + A_T \]

E.R. \( (A_{\text{CN}} < 170) \)

E.R. + F \( (220 > A_{\text{CN}} > 170) \)

F \( (A_{\text{CN}} > 220) \)

courtesy: Felipe Canto
Inter-nucleus potential

Two forces:
1. Coulomb force
   Long range, repulsive
2. Nuclear force
   Short range, attractive

Potential barrier due to the compensation between the two (Coulomb barrier)

- above barrier
- sub-barrier
- deep sub-barrier
Why subbarrier fusion?

Two obvious reasons:

- discovering new elements (SHE by cold fusion reactions)
- nuclear astrophysics (fusion in stars)
Why subbarrier fusion?

Two obvious reasons:

- discovering new elements (SHE)
- nuclear astrophysics (fusion in stars)

Other reasons:

- reaction mechanism
  strong interplay between reaction and structure
  (channel coupling effects)
  cf. high $E$ reactions: much simpler reaction mechanism
- many-particle tunneling
  cf. alpha decay: fixed energy
  tunneling in atomic collision: less variety of intrinsic motions
Basic of nuclear reactions

Shape, interaction, and excitation structures of nuclei scatting expt.
cf. Experiment by Rutherford ($\alpha$ scatt.)

Notation

$X(a, b)Y$

Projectile (beam)

Target nucleus

$^{208}\text{Pb}(^{16}\text{O}, ^{16}\text{O})^{208}\text{Pb}$ : $^{16}\text{O} + ^{208}\text{Pb}$ elastic scattering

$^{208}\text{Pb}(^{16}\text{O}, ^{16}\text{O'})^{208}\text{Pb}$ : $^{16}\text{O} + ^{208}\text{Pb}$ inelastic scattering

$^{208}\text{Pb}(^{17}\text{O}, ^{16}\text{O})^{209}\text{Pb}$ : 1 neutron transfer reaction

Notation:

$X(a, b)Y$
Scattering Amplitude

\[ \psi(r) \rightarrow e^{i k \cdot r} + f(\theta) \frac{e^{i k r}}{r} \]

\[ = (\text{incident wave}) + (\text{scattering wave}) \]
Differential cross section

\[
\psi_{sc}(r) \sim f(\theta) \frac{e^{ikr}}{r}
\]

The number of scattered particle through the solid angle of \(d\Omega\) per unit time:

\[
N_{\text{scatt}} = \mathbf{j}_{sc} \cdot \mathbf{e}_r \ r^2 d\Omega
\]

\[
\mathbf{j}_{sc} = \frac{\hbar}{2im} \left[ \psi_{sc}^* \nabla \psi_{sc} - \text{c.c.} \right] \sim \frac{k\hbar}{m} \frac{|f(\theta)|^2}{r^2} \mathbf{e}_r
\]

(flux for the scatt. wave)

\[
\frac{d\sigma}{d\Omega} = |f(\theta)|^2
\]
Scattering Amplitude

Motion of Free particle:

\[-\frac{\hbar^2}{2m} \nabla^2 \psi = E \psi = \frac{k^2 \hbar^2}{2m} \psi\]

\[
\psi(r) = e^{ik \cdot r} = \sum_{l=0}^{\infty} (2l + 1) i^l j_l(kr) P_l(\cos \theta)
\]

\[
\rightarrow \frac{i}{2kr} \sum_{l=0}^{\infty} (2l + 1) i^l \left[ e^{-i(kr-l\pi/2)} - e^{i(kr-l\pi/2)} \right] P_l(\cos \theta)
\]

In the presence of a potential:

\[
\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) - E \right] \psi = 0
\]

Asymptotic form of wave function

\[
\psi(r) \rightarrow \frac{i}{2kr} \sum_{l=0}^{\infty} (2l + 1) i^l \left[ e^{-i(kr-l\pi/2)} - S_l e^{i(kr-l\pi/2)} \right] P_l(\cos \theta)
\]

\[
= e^{ik \cdot r} + \left[ \sum_{l} (2l + 1) \frac{S_l - 1}{2ik} P_l(\cos \theta) \right] \frac{e^{ikr}}{r}
\]

\[f(\theta) \quad \text{(scattering amplitude)}\]
If only elastic scattering:

\[ |S_l| = 1 \quad \text{(flux conservation)} \]

\[ S_l = e^{2i\delta_l} \]

\[ \delta_l : \text{phase shift} \]
Optical potential and Absorption cross section

Reaction processes
- Elastic scatt.
- Inelastic scatt.
- Transfer reaction
- Compound nucleus formation (fusion)

Optical potential
\[
V_{\text{opt}}(r) = V(r) - iW(r) \quad (W > 0)
\]

\[\nabla \cdot j = \cdots = -\frac{2}{\hbar} W |\psi|^2\]

(note) Gauss’s law
\[
\int_S j \cdot n \, dS = \int_V \nabla \cdot j \, dV
\]
\[ \psi(r) \to \frac{i}{2k} \sum_l (2l + 1) i^l \frac{1}{r} [e^{-i(kr-l\pi/2)} - S_l e^{i(kr-l\pi/2)}] P_l(\cos \theta) \]

**Total incoming flux**

![Image of incoming flux arrows]

\[ j_{\text{in}}^{\text{net}} = \frac{k \hbar}{m} \cdot \frac{\pi}{k^2} \sum_l (2l + 1) \]

**Net flux loss:**

\[ j_{\text{in}}^{\text{net}} - j_{\text{out}}^{\text{net}} = \frac{k \hbar}{m} \cdot \frac{\pi}{k^2} \sum_l (2l + 1)(1 - |S_l|^2) \]

**Total outgoing flux**

![Image of outgoing flux arrows]

\[ j_{\text{out}}^{\text{net}} = \frac{k \hbar}{m} \cdot \frac{\pi}{k^2} \sum_l (2l + 1)|S_l|^2 \]

**Absorption cross section:**

\[ \sigma_{\text{abs}} = \frac{\pi}{k^2} \sum_l (2l + 1)(1 - |S_l|^2) \]
In the case of three-dimensional spherical potential:

\[
\psi(r) \rightarrow \frac{i}{2k} \sum_l (2l + 1) i^l \frac{1}{r} \left[ e^{-i(kr-l\pi/2)} - S_l e^{i(kr-l\pi/2)} \right] P_l(\cos \theta)
\]

\[ -S_l \sim R \quad \text{(reflection coeff.)} \quad \Rightarrow \quad P = |T|^2 = 1 - |S_l|^2 \]

\[
\sigma_{\text{abs}} = \frac{\pi}{k^2} \sum_l (2l + 1)(1 - |S_l|^2) = \frac{\pi}{k^2} \sum_l (2l + 1) P_l
\]
Overview of heavy-ion reactions

Heavy-ion: Nuclei heavier than $^4$He

Two forces:
1. Coulomb force
   Long range, repulsive
2. Nuclear force
   Short range, attractive

Potential barrier due to the compensation between these two (Coulomb barrier)
- Double Folding Potential

\[ V_{DF}(r) = \int dr_1 dr_2 \rho_1(r_1) \rho_2(r_2) \times v_{nn}(r + r_2 - r_1) \]

cf. Michigan 3 range Yukawa (M3Y) interaction

\[ v_{nn}(r) = 7999 \frac{e^{-4r}}{4r} - 2134 \frac{e^{-2.5r}}{2.5r} - 276 \delta(r) \text{ (MeV)} \]

- Phenomenological potential

\[ V_{WS}(r) = -\frac{V_0}{1 + \exp[(r - R_0)/a]} \]

\[ \rho(r) \sim \frac{\rho_0}{1 + \exp[(r - R_d)/a_d]} \]

\[ a_d \sim 0.54 \text{ (fm)} \]

\[ a \sim 0.63 \text{ (fm)} \]
Three important features of heavy-ion reactions

1. Coulomb interaction: important
2. Reduced mass: large
   \[ \mu = \frac{m_T m_P}{m_T + m_P} \]  
   (semi-) classical picture
   concept of trajectory
3. Strong absorption inside the Coul. barrier

**Automatic compound nucleus formation once touched (assumption of strong absorption)**
Strong absorption

the region of large overlap

- High level density (CN)
- Huge number of d.o.f.

Relative energy is quickly lost and converted to internal energy

Formation of hot CN (fusion reaction)
Partial decomposition of reaction cross section

Figure 4.18 Schematic decomposition of the total heavy-ion reaction cross section into contributions from different partial waves when (a) the grazing angular momentum (quantum number $\ell_g$) is below the critical angular momentum (quantum number $\ell_c$) that can be carried by the compound nucleus, and (b) when $\ell_g$ exceeds $\ell_c$. In both (a) and (b) the straight line is obtained from Equation (4.3) and the dashed areas indicate regions in which different types of heavy-ion nuclear reaction mechanisms predominate.

Taken from J.S. Lilley, "Nuclear Physics"
Classical Model for heavy-ion fusion reactions

Strong absorption

$l < l_g$ : can access to the strong absorption region classically

\[
\sigma^{cl} = 2\pi \int_0^{b_g} b \, db = \pi \, b_g^2
\]

\[
l_{cl} = k b \quad k = \sqrt{2\mu E/\hbar^2}
\]

\[
b_g = l_g / k
\]

\[
V_b + \frac{(kb_g)^2 \hbar^2}{2\mu R_b^2} = E
\]

\[
\sigma^{cl}(E) = \pi R_b^2 \left(1 - \frac{V_b}{E}\right)
\]


\[ \sigma_{\text{fus}}^{cl}(E) = \pi R_b^2 \left( 1 - \frac{V_b}{E} \right) \]

Classical fusion cross section is proportional to \( 1 / E \)

Taken from J.S. Lilley, “Nuclear Physics”
Fusion reaction and Quantum Tunneling

\[ \sigma_{\text{fus}}(E) = \frac{\pi}{k^2} \sum_l (2l + 1) P_l(E) \]

Fusion takes place by quantum tunneling at low energies!

Automatic CN formation once touched (assumption of strong absorption)

Probability of fusion = prob. to access to \( r_{\text{touch}} \)

Penetrability of barrier
**Quantum Tunneling Phenomena**

\[ T e^{-ikx} \]

\[ e^{-ikx} \]

\[ R e^{ikx} \]

\[ V(x) \]

\[ V_0 \]

\[ \psi(x) = T e^{-ikx} \quad (x \leq -a) \]

\[ = A e^{-\kappa x} + B e^{\kappa x} \quad (-a < x < a) \]

\[ = e^{-ikx} + R e^{ikx} \quad (x \geq a) \]

\[ k = \sqrt{2mE/\hbar^2} \]

\[ \kappa = \sqrt{2m(V_0 - E)/\hbar^2} \]

Tunnel probability: \[ P(E) = |T|^2 \]
For a parabolic barrier:

\[ V(x) = V_b - \frac{1}{2}m\Omega^2 x^2 \]

\[ P(E) = \frac{1}{1 + \exp \left[ \frac{2\pi}{\hbar\Omega} (V_b - E) \right]} \]

\[ \sim \frac{\hbar\Omega}{\hbar\Omega} \]
Energy derivative of penetrability

\[ P(E) = \theta(E - V_b) \]

\[ \frac{\hbar \Omega}{dP}{dE} = \delta(E - V_b) \]

(note) Classical limit
Potential Model: its success and failure

\[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V(r) + \frac{l(l + 1)}{2\mu r^2} - E\] \[u_l(r) = 0\]

Asymptotic boundary condition: \[u_l(r) \to H_l^{-}(kr) - S_l H_l^{+}(kr)\]

Fusion cross section:

\[\sigma_{\text{fus}} = \frac{\pi}{k^2} \sum_l (2l + 1) P_l\]

Mean angular mom. of CN:

\[\langle l \rangle = \frac{\sum_l l(2l + 1) P_l}{\sum_l (2l + 1) P_l}\]

\[P_l = 1 - |S_l|^2\]
Fusion cross section:

\[ \sigma_{\text{fus}} = \frac{\pi}{k^2} \sum_l (2l + 1) P_l \]

Mean angular mom. of CN:

\[ \langle l \rangle = \frac{\sum_l l(2l + 1) P_l}{\sum_l (2l + 1) P_l} \]
Wong’s formula

$$\sigma_{\text{ fus}}(E) = \frac{\pi}{k^2} \sum_l (2l + 1) P_l(E)$$

i) Approximate the Coul. barrier by a parabola:

$$P_0(E) = \frac{1}{1 + \exp\left[\frac{2\pi}{\hbar\Omega}(V_b - E)\right]}$$

ii) Approximate $P_l$ by $P_0$:

$$P_l(E) \sim P_0 \left( E - \frac{l(l + 1)\hbar^2}{2\mu R_b^2} \right)$$

(assume $l$-independent Rb and curvature)

iii) Replace the sum of $l$ with an integral

$$V(r) \sim V_b - \frac{1}{2} \mu \Omega^2 r^2$$

\text{C.Y. Wong, Phys. Rev. Lett. 31 (’73) 766}
\[ \sigma_{\text{fus}}(E) = \frac{\hbar \Omega}{2E} R_b^2 \log \left[ 1 + \exp \left( \frac{2\pi}{\hbar \Omega} (E - V_b) \right) \right] \]

(note) For \( E \gg V_b \)

\[ 1 \ll \exp \left( \frac{2\pi}{\hbar \Omega} (E - V_b) \right) \]

\[ \Rightarrow \sigma_{\text{fus}}(E) \sim \pi R_b^2 \left( 1 - \frac{V_b}{E} \right) = \sigma_{\text{fus}}^{cl}(E) \]

(note)

\[ \frac{d(E \sigma_{\text{fus}})}{dE} = \frac{\pi R_b^2}{1 + \exp \left[ \frac{2\pi}{\hbar \Omega} (V_b - E) \right]} = \pi R_b^2 \cdot P_{l=0}(E) \]
\[ \sigma_{\text{fus}}(E) = \frac{\hbar \Omega}{2E} R_b^2 \log \left[ 1 + \exp \left( \frac{2\pi}{\hbar \Omega} (E - V_b) \right) \right] \]
Comparison between prediction of pot. model with expt. data

Fusion cross sections calculated with a static energy independent potential

- Works well for relatively light systems
- Underpredicts $\sigma_{\text{fus}}$ for heavy systems at low energies

Potential model: Reproduces the data reasonably well for $E > V_b$

Underpredicts $\sigma_{\text{fus}}$ for $E < V_b$

What is the origin?

Inter-nuclear Potential is poorly parametrized?

Other origins?

cf. seminal work:
R.G. Stokstad et al., PRL41(‘78)465
PRC21(‘80)2427
With a deeper nuclear potential (but still within a potential model)....
Potential Inversion

\[ P_0(E) = \frac{1}{\pi R_b^2} \frac{d(E\sigma_{\text{fus}})}{dE} \]

(note)

\[ P_0(E) = \frac{1}{1 + S_0(E)}, \quad S_0(E) = \int_{r_1}^{r_2} dr \sqrt{\frac{2\mu}{\hbar^2}(V(r) - E)} \]

\[ t(E) \equiv r_2 - r_1 = -\frac{2}{\pi} \sqrt{\frac{\hbar^2}{2\mu}} \int_{E'} V_b \frac{dS_0(E')}{\sqrt{E' - E}} dE' \]

\[ \sigma_{\text{fus}} \rightarrow V(r) \]

Semi-classical app.

- Energy independent
- Local
- Single-ch.
- Potential inversion

\[ \sigma_{\text{fus}} \rightarrow V(r) \]

Semi-classical app.

- Energy independent
- Local
- Single-ch.

Unphysical potentials

Beautiful demonstration of C.C. effects

A.B. Balantekin, S.E. Koonin, and J.W. Negele, PRC28(’83)1565
- Potential inversion

\[ t(E) = 3 \pm 0.2 \text{ fm} \]

double valued potential (unphysical !!)
Potential model:
Reproduces the data reasonably well for $E > V_b$
Underpredicts $\sigma_{\text{fus}}$ for $E < V_b$

What is the origin?
- Inter-nuclear Potential is poorly parametrized?
- Other origins?
Target dependence of fusion cross section

Strong target dependence at $E < V_b$
Low-lying collective excitations in atomic nuclei

Low-lying excited states in even-even nuclei are collective excitations, and strongly reflect the pairing correlation and shell structure.

\[
\begin{array}{cccc}
E_4 / E_2 < 2 & E_4 / E_2 \sim 2 - 2.2 & E_4 / E_2 \sim 2.7 & E_4 / E_2 \sim 3.33 \\
2^+ & 2^+ & 0^+ & 0^+_B \\
4^+ & & & 2^+_\gamma \\
0^+ & & 4^+ & 6^+ \\
2^+ & & 2^+ & \\
0^+ & 2^+ & 4^+ & 2^+ \\
0^+_B & & & 0^+ \\
\text{"Shell Model" nucleus} & \text{Vibrator} & \text{Transitional} & \text{Rotor} \\
\end{array}
\]

SCHEMATIC EVOLUTION OF STRUCTURE
NEAR CLOSED – SHELL → MID SHELL

Taken from R.F. Casten, “Nuclear Structure from a Simple Perspective”
図3-4 Dyアイソトープの低励起スペクトル。励起エネルギーの単位はkeV。
Effect of collective excitation on $\sigma_{\text{fus}}$: rotational case

Excitation spectra of $^{154}\text{Sm}$

$0.903 \quad 8^+$
$0.544 \quad 6^+$
$0.267 \quad 4^+$
$0.082 \quad 2^+$
$0.0 \quad 0^+$

$^{154}\text{Sm}$ is deformed

$E_I \sim \frac{I(I+1)\hbar^2}{2\mathcal{J}}$

cf. Rotational energy of a rigid body
(Classical mechanics)

$$E = \frac{1}{2} I \omega^2 = \frac{I^2}{2\mathcal{J}}$$

($I = \mathcal{J}\omega$, $\omega = \dot{\theta}$)
Effect of collective excitation on $\sigma_{\text{fus}}$: rotational case

Comparison of energy scales

- Tunneling motion: $E_{\text{tun}} \sim \hbar \Omega \sim 3.5 \text{ MeV}$ (barrier curvature)
- Rotational motion: $E_{\text{rot}} \sim E_{2^+} \sim 0.08 \text{ MeV}$

$$V(r) \sim V_b - \frac{1}{2}\mu\Omega^2 r^2$$

$$E_{\text{tun}} \gg E_{\text{rot}} = I(I+1)\hbar^2/2\mathcal{J} \to 0$$

$$\mathcal{J} \to \infty$$

The orientation angle of $^{154}\text{Sm}$ does not change much during fusion

(note)

Ground state ($0^+$ state) when reaction starts

Mixing of all orientations with an equal weight

$$\sigma_{\text{fus}}(E) = \int_0^1 d(\cos \theta)\sigma_{\text{fus}}(E; \theta)$$
The orientation angle of $^{154}$Sm does not change much during fusion.

$$\sigma_{\text{fus}}(E) = \int_0^1 d(\cos \theta) \sigma_{\text{fus}}(E; \theta)$$

$V(r, \theta)$
The barrier is lowered for $\theta=0$ because an attraction works from large distances.

The barrier increases for $\theta=\pi/2$. because the rel. distance has to get small for the attraction to work

Def. Effect: enhances $\sigma_{\text{fus}}$ by a factor of $10 \sim 100$

Fusion: interesting probe for nuclear structure
Two effects of channel couplings

✓ energy loss due to inelastic excitations

✓ dynamical modification of the Coulomb barrier

large enhancement of fusion cross sections

cf. 2-level model: Dasso, Landowne, and Winther, NPA405(‘83)381
\[ \sigma_{\text{fus}}(E) = \int_0^1 d(\cos \theta) \sigma_{\text{fus}}(E; \theta) \]

One warning:
Don’t use this formula for light deformed nuclei, e.g., $^{28}\text{Si}$

\begin{align*}
4^+ & \quad 4.618 \text{ (MeV)} \\
2^+ & \quad 1.779 \text{ (MeV)} \\
0^+ & \quad 0
\end{align*}

\begin{align*}
^{28}\text{Si} + ^{208}\text{Pb}
\end{align*}
More quantal treatment: Coupled-Channels method

Coupling between rel. and intrinsic motions

\[ H = -\frac{\hbar^2}{2\mu} \nabla^2 + V_0(r) + H_0(\xi) + V_{\text{coup}}(r, \xi) \]

Entrance channel

Excited channel

\[ H_0(\xi) \phi_k(\xi) = \epsilon_k \phi_k(\xi) \]

\[ \Psi(r, \xi) = \sum_k \psi_k(r) \phi_k(\xi) \]
\[
H = -\frac{\hbar^2}{2\mu} \nabla^2 + V_0(r) + H_0(\xi) + V_{\text{coup}}(r, \xi)
\]

\[
\psi(r, \xi) = \sum_k \psi_k(r) \phi_k(\xi)
\]

\[
H_0(\xi) \phi_k(\xi) = \epsilon_k \phi_k(\xi)
\]

Schroedinger equation: \((H - E)\psi(r, \xi) = 0\)

\[
\langle \phi_k | H - E | \psi \rangle = 0
\]

or

\[
\left[ -\frac{\hbar^2}{2\mu} \nabla^2 + V_0(r) + \epsilon_k - E \right] \psi_k(r) + \sum_{k'} \langle \phi_k | V_{\text{coup}} | \phi_{k'} \rangle \psi_{k'}(r) = 0
\]

Coupled-channels equations
Coupled-channels equations

\[
-\frac{\hbar^2}{2\mu} \nabla^2 + V_0(r) + \epsilon_k - E \right] \psi_k(r) + \sum_{k'} \langle \phi_k | V_{\text{coup}} | \phi_{k'} \rangle \psi_{k'}(r) = 0
\]

equation for \( \psi_k \)

transition from \( \phi_k \) to \( \phi_{k'} \)

boundary condition:

\[
\psi_n(r) \rightarrow e^{-ik_0r} - S_0 e^{ik_0r} \quad (n = 0)
\]
\[
- S_n e^{ik_nr} \quad (n \neq 0)
\]

\[
P(E) = 1 - \sum_n |S_n|^2
\]

\[
k_n = \sqrt{2\mu(E - \epsilon_n)/\hbar^2}
\]
Angular momentum coupling

\[ H_0(\xi) \phi_{nI}m_I(\xi) = \epsilon_{nI} \phi_{nI}m_I(\xi) \]

Total ang. mom.: \[ I + l = J \]

\[ \psi(r, \xi) = \sum_k \psi_k(r)\phi_k(\xi) = \sum_{n,l,I} \frac{u_{nlI}(r)}{r} [Y_l(\hat{r})\phi_{nI}(\xi)](JM) \]

\[ \langle [Y_l\phi_{nI}]^{(JM)}|H - E|\psi \rangle = 0 \]

\[ \left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} + V_0(r) - E + \epsilon_{nI} \right] u_{nlI}(r) \]

\[ + \sum_{n'l'I'} \langle [Y_{l'}\phi_{n'I'}]^{(JM)}|V_{\text{coup}}(r)|[Y_l\phi_{nI}]^{(JM)} \rangle u_{n'l'I'}(r) = 0 \]
Boundary condition (with ang. mom. coupling)

\[ \psi(r, \xi) = \sum_{n,l,I} \frac{u_{nlI}(r)}{r} [Y_l(\hat{r}) \phi_{nI}(\xi)]^{(JM)} \]

\[ u_{nlI}(r) \to H_l^{(-)}(k_{nlI}r) \delta_{n,n_i} \delta_{l,l_i} \delta_{I,I_i} - \sqrt{\frac{k_0}{k_{nlI}}} S_{nlI} H_l^{(+)}(k_{nlI}r) \]

\[ P_l(E) = 1 - \sum_{nI} |S_{nlI}|^2 \]

\[ \sigma_{\text{fus}}(E) = \frac{\pi}{k^2} \sum_l (2l + 1) P_l(E) \]
Excitation structure of atomic nuclei

Excite the target nucleus via collision with the projectile nucleus

How does the target respond to the interaction with the projectile?

Standard approach: analysis with the coupled-channels method

- Inelastic cross sections
- Elastic cross sections
- Fusion cross sections

S-matrix $S_{nll}$
How to perform coupled-channels calculations?

1. Modeling: selection of excited states to be included

S. Raman et al., PRC43(‘91)521

low-lying collective states only

$^{116}\text{Sn}$
typical excitation spectrum: electron scattering data

low-lying collective excitations

GDR/GQR

low-lying non-collective excitations

- Giant Resonances: high $E_x$, smooth mass number dependence
  → adiabatic potential renormalization

- Low-lying collective excitations: barrier distributions, strong isotope dependence

- Non-collective excitations: either neglected completely or implicitly treated through an absorptive potential

$E_{\text{GDR}} \sim 79A^{-1/3}$ MeV
$E_{\text{GQR}} \sim 65A^{-1/3}$ MeV

M. Sasao and Y. Torizuka, PRC15('77)217
2. Nature of collective states: vibration? or rotation?

a) Vibrational coupling

excitation operator: \( \hat{O} = \frac{\beta}{\sqrt{4\pi}} (a + a^\dagger) \)

\[
\begin{align*}
0^+, 2^+, 4^+ & \quad \frac{(a^\dagger)^2}{\sqrt{2}} |0\rangle \\
2^+ & \quad a^\dagger |0\rangle \\
0^+ & \quad |0\rangle \\
\end{align*}
\]

\[
|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle \\
\epsilon_n = (n + 1/2) \hbar \omega
\]

\[
\langle n | O | n' \rangle = \frac{\beta}{\sqrt{4\pi}} \left( \sqrt{n'} \delta_{n,n'-1} + \sqrt{n' + 1} \delta_{n,n'+1} \right)
\]

\[
= \begin{pmatrix} 0 & F & 0 \\ F & \epsilon & \sqrt{2}F \\ 0 & \sqrt{2}F & 2\epsilon \end{pmatrix}
\]
Vibrational excitations

Bethe-Weizacker formula: Mass formula based on Liquid-Drop Model

\[ B(N, Z) = a_v A - a_s A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_{\text{sym}} \frac{(N - Z)^2}{A} \]

For a deformed shape,

\[ a = R \cdot (1 + \epsilon) \]
\[ b = R \cdot (1 + \epsilon)^{-1/2} \]

\[ E_{\text{surf}} = E_{\text{surf}}^{(0)} (1 + 2\epsilon^2/5 + \cdots) \]
\[ E_C = E_C^{(0)} (1 - \epsilon^2/5 + \cdots) \]
In general \[ R(\theta, \phi) = R_0 \left( 1 + \sum_{\lambda, \mu} \alpha_{\lambda \mu} Y_{\lambda \mu}^* \right) \]

\[ V = \frac{1}{2} \sum_{\lambda, \mu} C_\lambda |\alpha_{\lambda \mu}|^2 \]

Harmonic oscillation

\[ \lambda=2: \text{Quadrupole vibration} \]

Movie: Dr. K. Arita (Nagoya Tech. U.)
http://www.phys.nitech.ac.jp/~arita/
In general

$$R(\theta, \phi) = R_0 \left(1 + \sum_{\lambda, \mu} \alpha_{\lambda\mu} Y_{\lambda\mu}^* \right)$$

$$V = \frac{1}{2} \sum_{\lambda, \mu} C_\lambda |\alpha_{\lambda\mu}|^2$$

Harmonic oscillation

$\lambda = 3$: Octupole vibration

Movie: Dr. K. Arita (Nagoya Tech. U.)
http://www.phys.nitech.ac.jp/~arita/
Double phonon states

\begin{align*}
4^+ & : 1.282 \text{ MeV} \\
2^+ & : 1.208 \text{ MeV} \\
0^+ & : 1.133 \text{ MeV} \\
2^+ & : 0.558 \text{ MeV}
\end{align*}

Microscopic description

Random phase approximation (RPA)

$^{114}\text{Cd}$
2. Nature of collective states: vibration? or rotation?

a) Vibrational coupling

excitation operator: \( \hat{O} = \frac{\beta}{\sqrt{4\pi}} (a + a^\dagger) \)

\[
\begin{align*}
|0^+, 2^+, 4^+\rangle & \rightarrow \left( a^\dagger \right)^2 \frac{1}{\sqrt{2}} |0\rangle \\
2^+ & \rightarrow a^\dagger |0\rangle \\
0^+ & \rightarrow |0\rangle
\end{align*}
\]

\[
\langle n | O | n' \rangle = \frac{\beta}{\sqrt{4\pi}} \left( \sqrt{n'} \delta_{n,n'-1} + \sqrt{n'+1} \delta_{n,n'+1} \right)
\]

\[
= \begin{pmatrix}
0 & F & 0 \\
F & \epsilon & \sqrt{2}F \\
0 & \sqrt{2}F & 2\epsilon
\end{pmatrix}
\]

\[
|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle \\
\epsilon_n = (n + 1/2) \hbar \omega
\]
2. Nature of collective states: vibration? or rotation?

b) Rotational coupling

excitation operator: \( \hat{O} = \beta Y_{20}(\theta) (+ \beta_4 Y_{40}(\theta) + \cdots) \)

\[ |I\rangle = |Y_{I0}\rangle \]

\[ \epsilon_I = \frac{I(I + 1)}{6} \cdot E_{I=2} \]

\[
\langle I | O | I' \rangle = \sqrt{\frac{5 \cdot (2I + 1)(2I' + 1)}{4\pi}} \left( \begin{array}{ccc} I & 2 & I' \\ 0 & 0 & 0 \end{array} \right)^2 \\
= \begin{pmatrix} 0 & F & 0 \\ F & \epsilon + \frac{2\sqrt{5}}{7}F & \frac{6}{7}F \frac{10\epsilon}{3} + \frac{20\sqrt{5}}{77}F \end{pmatrix} \]
Vibrational coupling

\[ \hat{O} = \frac{\beta}{\sqrt{4\pi}} (a + a^\dagger) \]

Rotational coupling

\[ \hat{O} = \beta Y_{20}(\theta) \]

0\(^+\), 2\(^+\), 4\(^+\)

\[ \begin{pmatrix}
  0 & F & 0 \\
  F & \epsilon & \sqrt{2}F \\
  0 & \sqrt{2}F & 2\epsilon \\
\end{pmatrix} \]

4\(^+\)

\[ \begin{pmatrix}
  0 & F \\
  F & \epsilon + \frac{2\sqrt{5}}{7}F \\
  \frac{6}{7}F & \frac{10}{3}\epsilon + \frac{20\sqrt{5}}{77}F \\
\end{pmatrix} \]

cf. reorientation term

\[ F = \frac{\beta}{\sqrt{4\pi}} \]
3. Coupling constants and coupling potentials

**Deformed Woods-Saxon model:**

\[
V_{WS}(r) = -\frac{V_0}{1 + \exp[(r - R_0)/a]}
\]

\[
= -\frac{V_0}{1 + \exp[(r - R_P - R_T)/a]}
\]

\[
R_T \rightarrow R_T \left(1 + \sum_{\mu} \alpha_{\lambda \mu} Y_{\lambda \mu}^*(\theta, \phi)\right)
\]

excitation operator

\[
V_{WS}(r) = -\frac{V_0}{1 + \exp[(r - R_0 - R_T \alpha_{\lambda} \cdot Y_{\lambda}(\vec{r}))]/a}
\]
Coupling Potential: Collective Model

\[ R(\theta, \phi) = R_T \left( 1 + \sum_{\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}^*(\theta, \phi) \right) \]

- **Vibrational case**

\[ \alpha_{\lambda\mu} = \frac{\beta_\lambda}{\sqrt{2\lambda + 1}} \left( a_{\lambda\mu}^\dagger + (-)^\mu a_{\lambda\mu} \right) \]

- **Rotational case**

Coordinate transformation to the body-fixed frame

\[ \alpha_{\lambda\mu} = \sqrt{\frac{4\pi}{2\lambda + 1}} \beta_\lambda Y_{\lambda\mu}(\theta_d, \phi_d) \quad \text{(for axial symmetry)} \]

In both cases

\[ \beta_\lambda = \frac{4\pi}{3Z_TR_T^\lambda} \sqrt{\frac{B(E\lambda)}{e^2}} \]

(note) coordinate transformation to the rotating frame \((\hat{r} = 0)\)

\[ \sum_\mu \alpha_{\lambda\mu} Y_{\lambda\mu}^*(\theta, \phi) \rightarrow \sqrt{\frac{2\lambda + 1}{4\pi}} \alpha_{\lambda0} \]
Deformed Woods-Saxon model (collective model)

CCFULL
K.H., N. Rowley, and A.T. Kruppa,
Comp. Phys. Comm. 123(’99)143

CCFULL Home Page
K. Hagino, N. Rowley, and A.T. Kruppa
A FORTRAN77 program for coupled-channels calculations with all order couplings for heavy-ion fusion reactions

- Publication

A program for coupled-channels calculations with all order couplings for heavy-ion fusion reactions

- Program (the latest version)
  Sample input and output files

- The original version published in CPC

- A version with two different modes of excitation both in the proj. and in the tanc. (but with a simple harmonic oscillator coupling)
  Sample input and output files

- A version with an imaginary potential
  Sample input and output files

http://www.nucl.phys.tohoku.ac.jp/~hagino/ccfull.html
i) all order couplings

\[ V_{\text{coup}}(r, \hat{O}) = V_{\text{coup}}^{(N)}(r, \hat{O}) + V_{\text{coup}}^{(C)}(r, \hat{O}) \]

**Nuclear coupling:**

\[ V_{\text{coup}}^{(N)}(r, \hat{O}) = -\frac{V_0}{1 + \exp[(r - R_0 - R_T\hat{O})/a]} \]

**Coulomb coupling:**

\[ V_{\text{coup}}^{(C)}(r, \hat{O}) = \frac{3}{2\lambda + 1} Z_P Z_T e^2 \frac{R_T^\lambda}{r^{\lambda+1}} \hat{O} \]
i) all order couplings

\[ V_{\text{coup}}^{(N)}(r, \hat{O}) = \frac{-V_0}{1 + \exp[(r - R_0 - R_T\hat{O})/a]} \]

\[ \sim V_N(r) - R_T\hat{O} \frac{dV_N(r)}{dr} \]
**CCFULL**


**i) all order couplings**

\[ V_{coup}^{(N)}(r, \hat{O}) = \frac{V_0}{1 + \exp[(r - R_0 - R_T\hat{O})/a]} \]

\[ \sim V_N(r) - R_T\hat{O} \frac{dV_N(r)}{dr} \]

K.H., N. Takigawa, M. Dasgupta, D.J. Hinde, and J.R. Leigh, PRC55(‘97)276
ii) isocentrifugal approximation

$\lambda$: independent of excitations

$V_{\text{coup}}(r, \xi) = f(r)Y_\lambda(\hat{r}) \cdot T_\lambda(\xi) \rightarrow \sqrt{2\lambda + 1}f(r)T_{\lambda_0}(\xi)$

"Spin-less system"
$^{16}\text{O} + ^{144}\text{Sm} (2^+)$

$E_{cm} = 65$ MeV

K.H. and N. Rowley, PRC69(’04)054610
iii) incoming wave boundary condition (IWBC)

\[ \sigma_{\text{ fus}} = \frac{\pi}{k^2} \sum_l (2l + 1) P_l \quad \text{(} P_l = 1 - |S_l|^2 \text{)} \]

(1) Complex potential

\[ V(r) = V_R(r) - iW(r) \]

(2) IWBC

\[ u_l(r) = T_l \exp \left( -i \int_{r_{\text{abs}}}^{r} k_l(r')dr' \right) \]

(Incoming Wave Boundary Condition)

\[ k_l(r) = \sqrt{2\mu/\hbar^2[E - V_R(r) - l(l+1)\hbar^2/2\mu r^2]} \]

- Only Real part of Potential
- More efficient at low energies \[ P_l = |T_l|^2 \]

\[ \text{cf. } |S_l| \sim 1 \text{ at low } E \]
http://www.nucl.phys.tohoku.ac.jp/~hagino/ccfull.html
<table>
<thead>
<tr>
<th>Value 1</th>
<th>Value 2</th>
<th>Value 3</th>
<th>Value 4</th>
<th>Value 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>16.0</td>
<td>8.0</td>
<td>144.0</td>
<td>62.0</td>
<td></td>
</tr>
<tr>
<td>1.20</td>
<td>-1.00</td>
<td>1.060</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>1.810</td>
<td>0.205</td>
<td>3.0</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>1.660</td>
<td>0.110</td>
<td>2.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>6.130</td>
<td>0.733</td>
<td>3.0</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>105.10</td>
<td>1.10</td>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>55.0</td>
<td>70.0</td>
<td>1.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30.0</td>
<td>0.05</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Reaction system
$(A_p=16, Z_p=8, A_t=144, Z_t=62)$
- $r_p$, $I_{vibrot_p}$, $r_t$, $I_{vibrot_t}$
- (inert projectile, and vib. for targ.)
reaction system
\((A_p=16, Z_p=8, A_t=144, Z_t=62)\)
\(r_p, Ivibrot_p, r_t, Ivibrot_t\)
(inert projectile, and vib. for targ.)

\[
V_{\text{COUP}}^{(N)}(r, \hat{O}) = -\frac{V_0}{1 + \exp[(r - R_0 - R_T\hat{O})/a]}
\]

\[
R_T = r_t A_t^{1/3} \quad \text{(fm)}
\]

If \(Ivibrot_t = 0\): \(O = O_{\text{vib}}\)
\(Ivibrot_t = 1\): \(O = O_{\text{rot}}\)
\(Ivibrot_t = -1\): \(O = 0\) (inert)

similar for the projectile
<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_p$ = 16, $Z_p$ = 8, $A_t$ = 144, $Z_t$ = 62</td>
<td></td>
</tr>
<tr>
<td>Inert projectile, and vib. for targ.</td>
<td></td>
</tr>
<tr>
<td>Properties of the targ. excitation</td>
<td></td>
</tr>
<tr>
<td>$E_{1st}$ = 1.81 MeV</td>
<td>1.81</td>
</tr>
<tr>
<td>$\beta = 0.205$</td>
<td>3-$^1_{44}$Sm</td>
</tr>
<tr>
<td>$\lambda = 3$</td>
<td>0+</td>
</tr>
<tr>
<td>$N_{phonon} = 1$</td>
<td></td>
</tr>
<tr>
<td>Coupling to 3- vibrational state in the target with def. parameter $\beta = 0.205$</td>
<td></td>
</tr>
</tbody>
</table>

\[ \alpha_{\lambda\mu} = \frac{\beta_{\lambda}}{\sqrt{2\lambda + 1}} \left( a_{\lambda\mu}^{\dagger} + (-)^{\mu} a_{\lambda\mu} \right) \]

\[ \beta_{\lambda} = \frac{4\pi}{3Z_TR_T^\lambda} \sqrt{\frac{B(E\lambda)}{e^2}} \]
ccfull.inp

(A_p=16, Z_p=8, A_t=144, Z_t=62)

(inert projectile, and vib. for targ.)

properties of the targ. excitation

\( E_{1st} = 1.81 \text{ MeV} \)
\( \beta = 0.205 \)
\( \lambda = 3 \)
\( N_{\text{phonon}} = 2 \)

\( 1.81 \times 2 \quad (3^-)^2 \)
\( \sqrt{2} \quad \beta_3 \)
\( 1.81 \quad 3^- \)
\( \beta_3 \)

0 \quad \text{Sm}^{144}_{0^+}
<table>
<thead>
<tr>
<th>E2+</th>
<th>β2</th>
<th>β4</th>
<th>Nrot</th>
</tr>
</thead>
<tbody>
<tr>
<td>6x7x0.08/6</td>
<td>6+</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4x5x0.08/6</td>
<td>4+</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.08</td>
<td>2+</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0+</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3 excited states (Nrot=3) + g.s.
(Ap=16, Zp=8, At=144, Zt=62)

(inert projectile, and vib. for targ.)

(properties of the targ. excitation)

(same as the previous line, but the second mode of excitation in the target nucleus (vibrational coupling only))

$N_{\text{phonon}} = 0 \Rightarrow$ no second mode
(Ap=16, Zp=8, At=144, Zt=62)  
(inert projectile, and vib. for targ.)

properties of the targ. excitation

second mode in the targ.

(note) if N_{phonon} = 1:  
the code will ask you while you run it whether your coupling scheme is (a) or (b)

\[ 1.81, 1.66 (a) \]
\[ 1.81, 1.66 (b) \]

\[ 1.81, 1.66 \]
\[ 3^- \rightarrow 2^+ \]

\[ 0, 0, 0.3 \]

\[ 105.1, 1.1, 0.75 \]

\[ 55.0, 70.0, 1.0 \]

\[ 30, 0.05 \]

\[ ^{144}\text{Sm} \]
### ccfull.inp

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
</table>
| 1.2 | -1 | 1.06 | 0 | (inert projectile, and vib. for targ.)
| 1.81 | 0.205 | 3 | 1 | properties of the targ. excitation
| 1.66 | 0.11 | 2 | 1 | second mode in the targ.
| 6.13 | 0.733 | 3 | 1 | properties of the proj. excitation
| 0 | 0 | 0.3 |   | (similar as the third line)
| 105.1 | 1.1 | 0.75 |   | (will be skipped for an inert projectile)
| 55. | 70. | 1. |   | 
| 30 | 0.05 |   |   |
ccfull.inp

(A_p=16, Z_p=8, A_t=144, Z_t=62)

(inert projectile, and vib. for targ.)

properties of the targ. excitation

second mode in the targ.

properties of the proj. excitation
(similar as the third line)

transfer coupling (g.s. to g.s.)

\[
Q_{tr} = +3 \text{ MeV}
\]

\[
F_{tr}(r) = F \frac{dV_N}{dr} \quad (A_p' + A_t')
\]

* no transfer coup. for \( F = 0 \)
ccfull.inp

\[ 16., 8., 144., 62. \]  \( \rightarrow \)  \( (A_p=16, \ Z_p=8, \ A_t=144, \ Z_t=62) \)

\[ 1.2, -1, 1.06, 0 \]  \( \rightarrow \)  \( \text{(inert projectile, and vib. for targ.)} \)

\[ 1.81, 0.205, 3, 1 \]  \( \rightarrow \)  \( \text{properties of the targ. excitation} \)

\[ 1.66, 0.11, 2, 1 \]  \( \rightarrow \)  \( \text{second mode in the targ.} \)

\[ 6.13, 0.733, 3, 1 \]  \( \rightarrow \)  \( \text{properties of the proj. excitation} \)

\( \text{(similar as the third line)} \)

\[ 0, 0., 0.3 \]  \( \rightarrow \)  \( \text{transfer coupling (g.s. to g.s.)} \)

\[ 105.1, 1.1, 0.75 \]  \( \rightarrow \)  \( \text{potential parameters} \)

\[ V_N(r) = -\frac{V_0}{1 + \exp[(r - R_0)/a]} \]

\[ V_0 = 105.1 \text{ MeV}, \ a = 0.75 \text{ fm} \]

\[ R_0 = 1.1 \times (A_p^{1/3} + A_t^{1/3}) \text{ fm} \]
<table>
<thead>
<tr>
<th>Properties</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inert projectile, and vib. for targ.</td>
<td>1.2, -1, 1.06, 0</td>
</tr>
<tr>
<td>Properties of the targ. excitation</td>
<td>1.81, 0.205, 3, 1</td>
</tr>
<tr>
<td>Second mode in the targ.</td>
<td>1.66, 0.11, 2, 1</td>
</tr>
<tr>
<td>Properties of the proj. excitation</td>
<td>6.13, 0.733, 3, 1</td>
</tr>
<tr>
<td>Similar as the third line</td>
<td></td>
</tr>
<tr>
<td>Transfer coupling (g.s. to g.s.)</td>
<td>0, 0, 0.3</td>
</tr>
<tr>
<td>Potential parameters</td>
<td>105.1, 1.1, 0.75</td>
</tr>
<tr>
<td>$E_{\text{min}}, E_{\text{max}}, \Delta E$ (c.m. energies)</td>
<td>55., 70., 1.</td>
</tr>
<tr>
<td>$R_{\text{max}}, \Delta r$</td>
<td>30, 0.05</td>
</tr>
</tbody>
</table>
16O + 144Sm Fusion reaction
-----------------------------------
Phonon Excitation in the targ.: beta_N= 0.205, beta_C= 0.205, 
r0= 1.06(fm), omega= 1.81(MeV), Lambda= 3, Nph= 1
-----------------------------------
Potential parameters: V0= 105.10(MeV), r0= 1.10(fm), 
a= 0.75(fm), power= 1.00
Uncoupled barrier: Rb=10.82(fm), Vb= 61.25(MeV), 
Curv=4.25(MeV)
-----------------------------------
Ecm (MeV)   sigma (mb)        <l>
-------------------------------------
55.00000      0.97449E-02     5.87031
56.00000              0.05489     5.94333
57.00000              0.28583     6.05134
58.00000              1.36500     6.19272
59.00000              5.84375     6.40451
...................
69.00000          427.60179    17.16365
70.00000          472.46037    18.08247

In addition, “cross.dat” : fusion cross sections only
Coupled-channels equations and barrier distribution

\[
\begin{align*}
\left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{J(J+1)\hbar^2}{2\mu r^2} + V_0(r) - E + \epsilon_n \right] u_n(r) & \\
+ \sum_{n'} \langle \phi_n | V_{\text{coup}}(r, \xi) | \phi_{n'} \rangle u_{n'}(r) &= 0
\end{align*}
\]

\[
\begin{align*}
\lim_{\epsilon_n \to 0} \quad \text{(Sudden limit)} \\
\lim_{\epsilon_n \to \infty} \quad \text{(Adiabatic limit)}
\end{align*}
\]

\[
P_J(E) = 1 - \sum_n |S_n|^2 \\
\sigma_{\text{fus}}(E) = \frac{\pi}{k^2} \sum_J (2J+1)P_J(E)
\]

Calculate $\sigma_{\text{fus}}$ by numerically solving the coupled-channels equations

Let us consider a limiting case in order to understand (interpret) the numerical results

\[
\begin{align*}
\epsilon_n & : \text{very large} \quad \text{(Adiabatic limit)} \\
\epsilon_n & = 0 \quad \text{(Sudden limit)}
\end{align*}
\]
Comparison of two time scales

spring on a board

static case: $mg \sin \theta = k \Delta l \Rightarrow \Delta l = \frac{mg \sin \theta}{k}$
move very **slowly**? or move **instantaneously**?
Comparison of two time scales

similar related example: spring on a moving board

move very slowly? or move instantaneously?

- keep the original length ($\Delta l = 0$) “sudden limit”
- always at the equilibrium length ($\Delta l = mg \sin \theta / k$) “adiabatic limit”
fast reaction

slow reaction

large fluctuation

+ small fluctuation around the adiabatic path
Two limiting cases: (i) adiabatic limit

\[ H = -\frac{\hbar^2}{2\mu} \nabla^2 + V_0(r) + H_0(\xi) + V_{\text{coup}}(r, \xi) \]

much slower rel. motion than the intrinsic motion

\[ \hbar \Omega \ll \epsilon \]

(Barrier curvature v.s. Intrinsic excitation energy)

\[ [H_0(\xi) + V_{\text{coup}}(r, \xi)] \varphi_0(\xi; r) = \epsilon_0(r) \varphi_0(\xi; r) \]

\[ H_0(\xi) + V_{\text{coup}}(r, \xi) \rightarrow \epsilon_0(r) \]
c.f. Born-Oppenheimer approximation for hydrogen molecule

\[ [T_R + T_r + V(r, R)] \psi(r, R) = E \psi(r, R) \]

1. Consider first the electron motion for a fixed \( R \)
\[ [T_r + V(r, R)] u_n(r; R) = \epsilon_n(R) u_n(r; R) \]

2. Minimize \( \epsilon_n(R) \) with respect to \( R \)
Or 2’. Consider the proton motion in a potential \( \epsilon_n(R) \)
\[ [T_R + \epsilon_n(R)] \phi_n(R) = E \phi_n(R) \]
When $\varepsilon$ is large,

$$H_0(\xi) + V_{\text{coup}}(r, \xi) \rightarrow \varepsilon_0(r)$$

where

$$[H_0(\xi) + V_{\text{coup}}(r, \xi)] \varphi_0(\xi; r) = \varepsilon_0(r) \varphi_0(\xi; r)$$

Fast intrinsic motion

**Adiabatic potential renormalization**

$$V_{\text{ad}}(r) = V_0(r) + \varepsilon_0(r)$$

Giant Resonances, $^{16}\text{O}(3^-)$ [6.31 MeV]
typical excitation spectrum: electron scattering data

low-lying collective excitations

GDR/GQR

M. Sasao and Y. Torizuka, PRC15(‘77)217

$E_{\text{GDR}} \sim 79A^{-1/3}\text{ MeV}$
$E_{\text{GQR}} \sim 65A^{-1/3}\text{ MeV}$

low-lying non-collective excitations

- Giant Resonances: high $E_x$, smooth mass number dependence
  → adiabatic potential renormalization
- Low-lying collective excitations: barrier distributions, strong isotope dependence
- Non-collective excitations: either neglected completely or implicitly treated through an absorptive potential
Two limiting cases: (ii) sudden limit

\[ \epsilon \rightarrow 0 \]
\[ \epsilon_I = I(I + 1) \hbar^2 / 2 \mathcal{J} \]
\[ \mathcal{J} \rightarrow \infty \]

\[ \sigma_{\text{fus}}(E) = \int_0^1 d(\cos \theta) \sigma_{\text{fus}}(E; \theta) \]

**Coupled-channels:**

\[
\begin{pmatrix}
0 & f(r) & 0 \\
 f(r) & \frac{2\sqrt{5}}{7} f(r) & \frac{6}{7} f(r) \\
 0 & \frac{6}{7} f(r) & \frac{20\sqrt{5}}{77} f(r)
\end{pmatrix}
\]
diagonalize

\[
\begin{pmatrix}
\lambda_1(r) & 0 & 0 \\
0 & \lambda_2(r) & 0 \\
0 & 0 & \lambda_3(r)
\end{pmatrix}
\]

\[ P(E) = \sum_i w_i P(E; V_0(r) + \lambda_i(r)) \]

**Slow intrinsic motion**

**Barrier Distribution**
Barrier distribution

\[ P(E) = \sum_i w_i P(E; V_0(r) + \lambda_i(r)) \]
Barrier distribution: understand the concept using a spin Hamiltonian

Hamiltonian (example 1):

\[
H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_0(x) + \hat{\sigma}_z \cdot V_s(x)
\]

\[
\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

For Spin-up

\[
T_1 e^{-i k x} | \uparrow \rangle \quad e^{-i k x} | \uparrow \rangle \quad R_1 e^{i k x} | \uparrow \rangle
\]

\[
V_1(x) = V_0(x) + V_s(x)
\]

For Spin-down

\[
T_2 e^{-i k x} | \downarrow \rangle \quad e^{-i k x} | \downarrow \rangle \quad R_2 e^{i k x} | \downarrow \rangle
\]

\[
V_2(x) = V_0(x) - V_s(x)
\]
\[ H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_0(x) + \hat{\sigma}_z \cdot V_1(x) \]

Wave function (general form)
\[ \psi(x) = \psi_1(x) | \uparrow \rangle + \psi_2(x) | \downarrow \rangle = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \]

The spin direction does not change during tunneling:

\[ P(E) = w_\uparrow P_1(E) + w_\downarrow P_2(E) \]
\[ w_\uparrow + w_\downarrow = 1 \]
$P(E) = w_{\uparrow} P_1(E) + w_{\downarrow} P_2(E)$

Tunneling prob. is a weighted sum of tunnel prob. for two barriers

\[
\begin{align*}
V_1(x) &= V_0(x) + V_s(x) \\
V_2(x) &= V_0(x) - V_s(x)
\end{align*}
\]

\[\hbar\Omega \cdot \frac{dP}{dE}\]
- Tunnel prob. is enhanced at $E < V_b$ and hindered $E > V_b$
- $dP/dE$ splits to two peaks "barrier distribution"
- The peak positions of $dP/dE$ correspond to each barrier height
- The height of each peak is proportional to the weight factor

\[
P(E) = w_\uparrow P_1(E) + w_\downarrow P_2(E)
\]
\[
\frac{dP}{dE} = w_\uparrow \frac{dP_1}{dE} + w_\downarrow \frac{dP_2}{dE}
\]
simple 2-level model (Dasso, Landowne, and Winther, NPA405(‘83)381)

\[
\begin{align*}
\left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V_l(r) + \begin{pmatrix} 0 & F \\ F & \epsilon \end{pmatrix} - E\right] \begin{pmatrix} u_0(r) \\ u_1(r) \end{pmatrix} &= 0
\end{align*}
\]

entrance channel

excited channel

\[
\begin{align*}
\left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V_l(r) + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} - E\right] \begin{pmatrix} \phi_0(r) \\ \phi_1(r) \end{pmatrix} &= 0
\end{align*}
\]

\[
\begin{align*}
\phi_0(r) &= \alpha \cdot u_0(r) + \beta \cdot u_1(r) \\
\phi_1(r) &= \beta \cdot u_0(r) - \alpha \cdot u_1(r)
\end{align*}
\]
simple 2-level model (Dasso, Landowne, and Winther, NPA405(‘83)381)

Fig. 1. Illustration of how channel coupling increases transmission at energies below the barrier and decreases it above. Parts (a) and (b) indicate the classical limits for no coupling and coupling, respectively, while parts (c) and (d) indicate how quantum mechanical effects modify the corresponding curves.
Sub-barrier Fusion and Barrier distribution method

\[ \sigma_{\text{fus}}(E) = \frac{\hbar \Omega}{2E} R_b^2 \log \left[ 1 + \exp \left( \frac{2\pi}{\hbar \Omega} (E - V_b) \right) \right] \]

\[ \frac{d(E\sigma_{\text{fus}})}{dE} = \frac{\pi R_b^2}{1 + \exp \left[ \frac{2\pi}{\hbar \Omega}(V_b - E) \right]} = \pi R_b^2 \cdot P_{l=0}(E) \]

\[ D_{\text{fus}}(E) \equiv \frac{d^2(E\sigma_{\text{fus}})}{dE^2} \sim \pi R_b^2 \frac{dP_{l=0}}{dE} \]

(Fusion barrier distribution)

N. Rowley, G.R. Satchler, P.H. Stelson, PLB254(’91)25
\[ \frac{d}{dE} [E\sigma_{\text{fus}}(E)] \propto P(E) \]
\[ \frac{d^2}{dE^2} [E\sigma_{\text{fus}}(E)] \propto \frac{dP}{dE} \]

centered on \( E = V_b \)

N. Rowley, G.R. Satchler, P.H. Stelson, PLB254(’91)25
Barrier distribution measurements

Fusion barrier distribution

\[ D_{\text{fus}}(E) = \frac{d^2(E \sigma)}{dE^2} \]

Needs high precision data in order for the 2\textsuperscript{nd} derivative to be meaningful.
Experimental Barrier Distribution

Requires high precision data

\[ \sigma_{\text{fus}}(E) = \int_0^1 d(\cos \theta_T) \sigma_{\text{fus}}(E; \theta_T) \]

M. Dasgupta et al.,
Annu. Rev. Nucl. Part. Sci. 48(’98)401
Investigate nuclear shape through barrier distribution

Nuclear shapes

\[ R(\theta) = R_0 \left( 1 + \beta_2 Y_{20}(\theta) + \beta_4 Y_{40}(\theta) + \cdots \right) \]

\[ R_0 = 5.9 \text{ (fm)}, \quad \beta_2 = 0.3 \]

\[ \beta_4 = 0 \quad \beta_4 = 0.1 \quad \beta_4 = -0.1 \]
By taking the barrier distribution, one can very clearly see the difference due to $\beta_4$!

Fusion as a quantum tunneling microscope for nuclei
Advantage of fusion barrier distribution

Fusion Cross sections

Very strong exponential energy dependence

Difficult to see differences due to details of nuclear structure

Plot cross sections in a different way: Fusion barrier distribution

\[ D_{\text{fus}}(E) = \frac{d^2(E\sigma)}{dE^2} \]

N. Rowley, G.R. Satchler, P.H. Stelson, PLB254(’91)25

Function which is sensitive to details of nuclear structure
Example for spherical vibrational system

$^{16}\text{O} + ^{144}\text{Sm}$

Quadrupole moment:
$Q(3^-) = -0.70 \pm 0.02\text{b}$
Barrier distribution

$^{16}\text{O} + ^{144}\text{Sm}$

K. Hagino, N. Takigawa, and S. Kuyucak,
PRL79(’97)2943
Coupling to excited states $\rightarrow$ distribution of potential barrier

multi-dimensional potential surface

single barrier $\rightarrow$ a collection of many barriers

\[ P(E) = P[E, V(r)] \]
\[ \rightarrow P(E) = \sum_{\alpha} w_{\alpha} P[E, V_{\alpha}(r)] \]

(intrinsic coordinate)
Representations of fusion cross sections

i) $\sigma_{\text{fus}}$ vs $1/E$ (~70’s)

Classical fusion cross section is proportional to $1/E$:

$$\sigma_{\text{fus}}^c(E) = \pi R_b^2 \left( 1 - \frac{V_b}{E} \right)$$

Taken from J.S. Lilley, “Nuclear Physics”
ii) barrier distribution (\(~90’s\) )

\[\sigma (\text{mb})\]

\[d^2 (E\sigma/dE^2) \text{ (mb/MeV)}\]

\[E_{cm} \text{ (MeV)}\]
iii) logarithmic derivative (~00’s)

\[ \sigma_{\text{fus}}(E) = \frac{\hbar \Omega}{2E} R_b^2 \log \left[ 1 + \exp \left( \frac{2\pi}{\hbar \Omega} (E - V_b) \right) \right] \]

\[ \sim \frac{\hbar \Omega}{2E} R_b^2 \exp \left( \frac{2\pi}{\hbar \Omega} (E - V_b) \right) \quad (E \ll V_b) \]

\[ \frac{d}{dE} \log(E\sigma) = \frac{(E\sigma)'}{E\sigma} = \frac{2\pi}{\hbar \Omega} \]

cf. \( D_{\text{fus}} = (E\sigma)'' \)


M. Dasgupta et al., PRL99(‘07) 192701
deep subbarrier hindrance of fusion cross sections

C.L. Jiang et al., PRL89(‘02)052701; PRL93(‘04)012701
mechanism of deep subbarrier hindrance: not yet been fully understood

how to model the dynamics after touching?

T. Ichikawa, K.H., A. Iwamoto, PRC75('07) 064612 & 057603
Quantum reflection and quasi-elastic scattering

In quantum mechanics, reflection occurs even at $E > V_b$

\[ P(E) + R(E) = 1 \]

Reflection prob. carries the same information as penetrability, and barrier distribution can be defined in terms of reflection prob.
Quasi-Elastic Scattering

A sum of all the reaction processes other than fusion (elastic + inelastic + transfer + ……)

Detect all the particles which reflect at the barrier and hit the detector

In case of a def. target………

\[
\begin{aligned}
\sigma_{\text{fus}}(E) &= \int_0^1 d(\cos \theta_T)\sigma_{\text{fus}}(E; \theta_T) \\
\sigma_{\text{qel}}(E, \theta) &= \sum_I \sigma(E, \theta) = \int_0^1 d(\cos \theta_T)\sigma_{\text{el}}(E, \theta; \theta_T)
\end{aligned}
\]
Subbarrier enhancement of fusion cross sections

\[ \sigma_{\text{fus}}(E) = \int_0^1 d(\cos \theta) \sigma_{\text{fus}}(E; \theta) \]

\[ \sigma_{\text{qel}}(E, \theta) = \int_0^1 d(\cos \theta_T) \sigma_{\text{el}}(E, \theta; \theta_T) \]

Quasi-elastic scattering (elastic + inelastic)
Quasi-elastic barrier distribution:

\[ \sigma_{\text{fus}}(E) = \int_0^1 d(\cos \theta_T)\sigma_{\text{fus}}(E; \theta_T) \]

\[ D_{\text{fus}}(E) = \frac{d^2(E\sigma_{\text{fus}})}{dE^2} \]

\[ \sigma_{\text{qel}}(E, \theta) = \sum_I \sigma(E, \theta) = \int_0^1 d(\cos \theta_T)\sigma_{\text{el}}(E, \theta; \theta_T) \]

Quasi-elastic barrier distribution:

\[ D_{\text{qel}}(E) = -\frac{d}{dE} \left( \frac{\sigma_{\text{qel}}(E, \pi)}{\sigma_R(E, \pi)} \right) \]

H. Timmers et al., NPA584('95)190

(note) Classical elastic cross section in the limit of strong Coulomb field:

\[ \sigma_{\text{el}}^{cl}(E, \pi) = \sigma_R(E, \pi)\theta(V_b - E) \]

\[ \frac{\sigma_{\text{el}}^{cl}(E, \pi)}{\sigma_R(E, \pi)} = \theta(V_b - E) = R(E) \]
Quasi-elastic test function

Classical elastic cross section (in the limit of a strong Coulomb):

\[ \sigma_{\text{el}}^c(E, \pi) = \sigma_R(E, \pi) \theta(V_b - E) \]

\[ \frac{\sigma_{\text{el}}^c(E, \pi)}{\sigma_R(E, \pi)} = \theta(V_b - E) = R(E) \]

\[ -\frac{d}{dE} \left( \frac{\sigma_{\text{el}}^c(E, \pi)}{\sigma_R(E, \pi)} \right) = \delta(E - V_b) \]

Nuclear effects \quad \text{Semi-classical perturbation theory}

\[ \frac{\sigma_{\text{el}}(E, \pi)}{\sigma_R(E, \pi)} \sim \left( 1 + \frac{V_N(r_c)}{k \alpha} \sqrt{\frac{2a\pi k \eta}{E}} \right) \cdot R(E) \]

S. Landowne and H.H. Wolter, NPA351(’81)171
K.H. and N. Rowley, PRC69(’04)054610
The peak position slightly deviates from $V_b$.

- Low energy tail
- Integral over $E$: unity
- Relatively narrow width

Quasi-elastic test function

$$G_{qel}(E) \equiv - \frac{d}{dE} \left( \frac{\sigma_{el}(E, \pi)}{\sigma_R(E, \pi)} \right)$$

Close analog to fusion b.d.
Scaling property

Expt.: impossible to perform at $\theta = \pi$

Relation among different $\theta$?

$\lambda = 0$ ($\theta = \pi$)

Effective energy:

\[
E_{\text{eff}} \sim E - \frac{\lambda_c^2 \hbar^2}{2 \mu r_c^2} \frac{\sin(\theta/2)}{1 + \sin(\theta/2)} \\
D_{\text{qel}}(E, \theta) \sim D_{\text{qel}}(E_{\text{eff}}, \pi)
\]

$\lambda_c = \eta \cot(\theta/2)$
\[ \sigma_{el} / \sigma_R \] vs. \( E_{cm} \) (MeV)

- \( \theta = 180 \text{ deg.} \)
- \( \theta = 160 \text{ deg.} \)
- \( \theta = 160 \text{ deg. (shifted)} \)

\[ G_{qel} \] (MeV\(^{-1}\)) vs. \( E_{cm} \) (MeV)

- \( \theta = 180 \text{ deg.} \)
- \( \theta = 140 \text{ deg.} \)
- \( \theta = 140 \text{ deg. (shifted)} \)
Comparison of $D_{\text{fus}}$ with $D_{\text{qel}}$

\[ D_{\text{fus}}(E) = \frac{d^2(E\sigma_{\text{fus}})}{dE^2} \]

\[ D_{\text{qel}}(E) = -\frac{d}{dE} \left( \frac{\sigma_{\text{qel}}(E, \pi)}{\sigma_R(E, \pi)} \right) \]

H. Timmers et al., NPA584(’95)190

A gross feature is similar to each other

K.H. and N. Rowley, PRC69(’04)054610
Experimental barrier distribution with QEL scattering

\[ D_{\text{qel}}(E) = -\frac{d}{dE} \left( \frac{\sigma_{\text{qel}}(E, \pi)}{\sigma_R(E, \pi)} \right) \]

\[ 70\text{Zn} + 208\text{Pb} \]

\[ 70\text{Zn} : E_2 = 0.885 \text{ MeV}, \; 2 \text{ phonon}, \; 208\text{Pb} : E_3 = 2.614 \text{ MeV}, \; 3 \text{ phonon} \]

Muhammad Zamrun F., K. H., S. Mitsuoka, and H. Ikezoe, PRC77(’08)034604.

Experimental Data: S. Mitsuoka et al., PRL99(’07)182701
Experimental advantages for $D_{qel}$

$$D_{qel}(E) = -\frac{d}{dE} \left( \frac{\sigma_{qel}(E, \pi)}{\sigma_R(E, \pi)} \right) \quad D_{fus}(E) = \frac{d^2(E\sigma_{fus})}{dE^2}$$

- less accuracy is required in the data ($1^{st}$ vs. $2^{nd}$ derivative)
- much easier to be measured

*Qel*: a sum of everything

$a$ very simple charged-particle detector

*Fusion*: requires a specialized recoil separator

to separate ER from the incident beam

ER + fission for heavy systems

- several effective energies can be measured at a single-beam energy

$E_{\text{eff}} = 2E \sin(\theta/2)/[1 + \sin(\theta/2)]$
Scattering processes:

- Double folding pot.
- Woods-Saxon ($a \sim 0.63$ fm)
  - Successful

Fusion process: not successful

$\rightarrow \quad a \sim 1.0$ fm required (if WS)

A. Mukherjee, D.J. Hinde, M. Dasgupta, K.H., et al., PRC75(07)044608
How reliable is the DFM/WS?

What is an optimum potential?

deduction of fusion barrier from exp. data? (model independent analysis?)
Quasi-elastic scattering at deep subbarrier energies?

K.H., T. Takehi, A.B. Balantekin, and N. Takigawa, PRC71(’05) 044612
K. Washiyama, K.H., M. Dasgupta, PRC73(’06) 034607

QEL at deep subbarrier energies: sensitive only to the surface region

\[
\frac{\sigma_{el}(E, \pi)}{\sigma_R(E, \pi)} \sim \left(1 + \frac{V_N(r_c)}{ka} \frac{\sqrt{2a\pi k\eta}}{E} \right) \cdot R(E)
\]

\[
\sim 1 + \frac{V_N(r_c)}{ka} \frac{\sqrt{2a\pi k\eta}}{E}
\]
Summary

Heavy-Ion Fusion Reactions around the Coulomb Barrier

✧ Fusion and quantum tunneling
  Fusion takes place by tunneling

✧ Basics of the Coupled-channels method
  Collective excitations during fusion

✧ Concept of Fusion barrier distribution
  Sensitive to nuclear structure
  \[ D_{\text{fus}}(E) = \frac{d^2(E\sigma_{\text{fus}})}{dE^2} \]

✧ Quasi-elastic scattering and quantum reflection
  Complementary to fusion

Computer program: CCFULL

http://www.nucl.phys.tohoku.ac.jp/~hagino/ccfull.html
References

Nuclear Reaction in general

- G.R. Satchler, “Direct Nuclear Reactions”
- G.R. Satchler, “Introduction to Nuclear Reaction”
- R.A. Broglia and A. Winther, “Heavy-Ion Reactions”
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- D.M. Brink, “Semi-classical method in nucleus-nucleus collisions”
- P. Frobrich and R. Lipperheide, “Theory of Nuclear Reactions”

Heavy-ion Fusion Reactions

- M. Dasgupta et al., Annu. Rev. Nucl. Part. Sci. 48(’98) 401
- A.B. Balantekin and N. Takigawa, Rev. Mod. Phys. 70(’98) 77
- Proc. of Fusion03, Prog. Theo. Phys. Suppl. 154(’04)
- Proc. of Fusion97, J. of Phys. G 23 (’97)
- Proc. of Fusion06, AIP, in press.
Hamiltonian (example 3): more general cases

\[
H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_0(x) - \epsilon \sigma_z + \hat{\sigma}_x \cdot F(x) \\
= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_0(x) + \begin{pmatrix} -\epsilon & F(x) \\ F(x) & \epsilon \end{pmatrix}
\]

Let \( U(x) \left( \begin{pmatrix} -\epsilon & F(x) \\ F(x) & \epsilon \end{pmatrix} \right) U^\dagger(x) = \begin{pmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{pmatrix} \)

\[ P(E) = \sum_i w_i(E) P(E; V_0(x) + \lambda_i(x)) \]

K.H., N. Takigawa, A.B. Balantekin, PRC56(’97)2104 \( w_i(E) \sim \text{constant} \)

(note) Adiabatic limit: \( \epsilon \rightarrow \infty \rightarrow w_i(E) = \delta_{i,0} \)